Sheaf Theory and Applications

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These are lecture notes compiled from the course 'Sheaf Theory and Applications' taught by Adam Brown at IST Austria during the Spring semester of 2020. The primary aim of this course is to provide a self-contained introduction to sheaf cohomology, with an emphasis on computable examples and applications (when possible).

We take this opportunity to make the obligatory disclaimer that this is a living document, insofar as we expect to add, revise, and adjust these notes as needed over time. We therefore ask for patience and forgiveness for any errors in the manuscript. We appreciate if any suggested changes are addressed to the first author.

Contents

1 Introduction

Lecture 1: Introduction and Penrose triangle

In these notes we will discuss category theory, sheaf theory, homological algebra, and sheaf cohomology, all with an emphasis on applications and building intuition from examples.

In this lecture we will give several brief examples, meant to provide some visual (and non-technical non-jargon filled) intuition for what this course is about (sheaves and sheaf cohomology), before we dive into category theory in Lecture 2.

What are sheaves? Sheaves are mathematical objects which are used to study continuous maps between topological spaces, $f : Y \to X$. We can view these sheaves as data structures on X which systematically record information about the topological structure of fibers $f^{-1}(x)$ for each point $x \in X$.

Why are sheaves useful in mathematics? Firstly, sheaf theory gives a very general framework which encompasses many classical techniques in topology and geometry. For example (perhaps a somewhat trivial example), we can study the (co)homology of a topological space X through understanding the fiber of the map $p : X \to \{\star\}$. As a slight generalization of this example, we can consider a sheaf which captures the (co)homology of fibers of a Morse function $f : M \to \mathbb{R}$. This example leads quite naturally to the study of persistent homology, which has seen numerous applications and developments in recent years. Secondly, it is our opinion that sheaf theory has been useful in mathematics in part because it formalizes an incredibly intuitive approach to scientific understanding, namely, the process of understanding global structures through local inquiry.

On a more technical level, sheaves are useful because they allow us to study topology using algebra. By algebra, we are usually referring to the theory of modules, rings, or groups:

Definition 1.0.1. An *abelian group* is a pair $(G, +)$ such that

1. G is a set
2. $+ : G \times G \to G$ is a map which is
   (a) associative, $(a + b) + c = a + (b + c)$
   (b) commutative, $a + b = b + a$
3. G has an identity ($\mathrm{id}_G \in G$ such that $a + \mathrm{id}_G = a$ for all $a \in G$) and inverses (for each $a \in G$, there exists $-a \in G$ such that $-a + a = \mathrm{id}_G$).

Example 1.0.2. 1. $(\mathbb{Z}, +)$, integers with addition

2. $(\mathbb{Z}/2\mathbb{Z}, +)$, the set $\{0, 1\}$ with $1 + 1 = 0$
3. $(U(1) = \{e^{i\theta}\}, \times)$
4. Nonexample: $(\mathbb{Z}, \times)$

Let X be a topological space.

Definition 1.0.3. A *sheaf on X* is a pair (Y, f) such that

1. Y is a topological space (not necessarily Hausdorff)
2. $f : Y \to X$ is a local homeomorphism onto X
3. each fiber $f^{-1}(x)$, $x \in X$, is endowed with the structure of an abelian group

4. the group operations

$$\begin{aligned} a+ : f^{-1}(x) &\to f^{-1}(x) \\ b &\mapsto a+b \end{aligned}$$

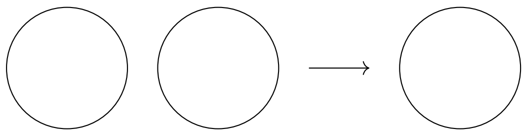
for each $a \in f^{-1}(x)$, and

$$\begin{aligned} - : f^{-1}(x) &\to f^{-1}(x) \\ b &\mapsto -b \end{aligned}$$

are continuous maps

Example 1.0.4. Orientation sheaf on S^1, $\omega_{S^1} = (Y, f)$:

$$\begin{aligned} f : S^1 \times \mathbb{Z}/2\mathbb{Z} &\to S^1 \\ (\theta, n) &\mapsto \theta \end{aligned}$$

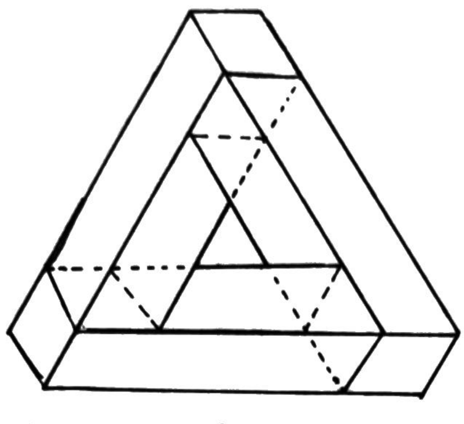


Definition 1.0.5. A *section* of $f : Y \to X$ over $U \subset X$ is a continuous map $s : U \to Y$ such that $f \circ s = \mathrm{id}_U$.

Exercise 1.0.6. *How many sections does ω_{S^1} have over S^1?*

One of the primary tools used in sheaf theory is sheaf cohomology. This cohomology generalizes several classical cohomology theories such as de Rham cohomology and Cech cohomology. Speaking very generally, sheaf cohomology is an invariant which quantifies obstructions to extending local sections of a given sheaf to global sections of the sheaf.

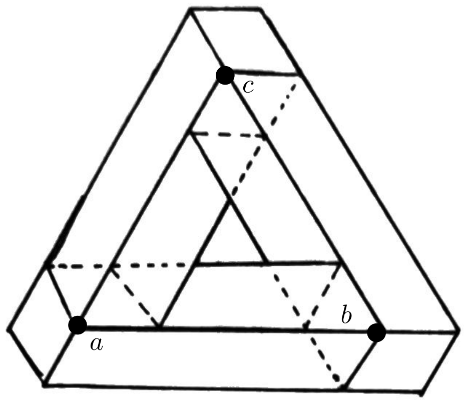
Example 1.0.7. In this example we will consider the Penrose triangle as a visualization of sheaf cohomology, following [Pen92].



We will solve three separate problems related to the above figure.

Exercise 1.0.8. *Show that the Penrose triangle is an impossible figure.*

Solution: We begin by measuring the distance from the observer to three points on the figure illustrated below.



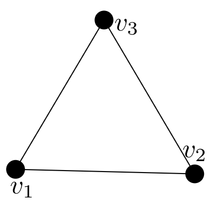
Let a, b, c denote the distance from the observer to the corresponding marked points on the figure above. The illustration suggests the following inequalities:

$$a < b$$
$$b < c$$
$$c < a.$$

The figure is an impossible shape because there are no solutions to the above system of inequalities.

Exercise 1.0.9. *Give an explicit example of a degree 1 cohomology class of S^1 (over $\mathbb{R}$).*

Solution: We can use a simplicial structure on S^1 to give a concrete description of cohomology classes. Let



be a triangulation of S^1, consisting of three 0-cells and three 1-cells. Then the cochain complex is given by

$$\mathbb{R}^3 \xrightarrow{\partial} \mathbb{R}^3$$
$$(x, y, z) \mapsto (x - y, y - z, x - z).$$

A representative of degree 1 cohomology is given by a triple (A, B, C) such that $A, B < 0$ and $C > 0$. Let (a, b, c) denote the triple from Exercise 1.0.8. Then $(a - b, b - c, a - c)$ is a representative of a degree 1 cohomology class of S^1.

Exercise 1.0.10. *Let $\omega_{S^1}(\mathbb{R}) = (Y, f)$ be the sheaf*

$$Y = S^1 \times \mathbb{R} \xrightarrow{f} S^1,$$

where $\mathbb{R}$ has the discrete topology. Let $\{U_1, U_2, U_3\}$ denote a cover of S^1 by connected, simply connected open sets with $U_1 \cap U_2 \cap U_3 = \varnothing$. Let s_i be a section of $\omega_{S^1}(\mathbb{R})$ over U_i.

1. *When is there a section s of $\omega_{S^1}(\mathbb{R})$ over S^1 which restricts to s_i for each i?*
2. *Suppose $s_{i,j}$ is a section of $\omega_{S^1}(\mathbb{R})$ over $U_i \cap U_j$, for $i < j$. Are there sections s_i of $\omega_{S^1}(\mathbb{R})$ over U_i for each i such that $s_{i,j} = s_i - s_j$?*

Solution: A section s_i over U_i is a constant function

$$s_i : U_i \to \mathbb{R},$$

i.e. $s_i \in \mathbb{R}$. There is a section s of $\omega_{S^1}(\mathbb{R})$ over S^1 which restricts to s_i for each i if and only if $s_1 = s_2 = s_3$, i.e. $(s_1, s_2, s_3) \in H^0(S^1, \mathbb{R})$.

To answer the second question, we notice that if $s_{1,2} < 0$, $s_{2,3} < 0$, and $s_{1,3} > 0$, then there is no combination of sections s_1, s_2, s_3 such that the required inequalities are satisfied. In other words, if $s_{1,2} < 0$, $s_{2,3} < 0$, and $s_{1,3} > 0$, then the triple $(s_{1,2}, s_{2,3}, s_{1,3})$ is a representative element of degree 1 *sheaf cohomology* of the sheaf $\omega_{S^1}(\mathbb{R})$.

One can notice that the impossibility of the Penrose triangle is closely related to cohomology classes of the circle, which is in turn closely related to questions about sections of $\omega_{S^1}(\mathbb{R})$, which is closely related to the (yet to be defined) sheaf cohomology of $\omega_{S^1}(\mathbb{R})$. This example is meant to provide some visual intuition for what sheaf cohomology measures (although we will not define sheaf cohomology for some time).

We now take a turn toward category theory in order to provide a useful framework for which to introduce and study sheaf theory and sheaf cohomology.

2 Category theory

Lecture 2: Categories and functors

References. [Cur14, Chapter 1], [Mac98, Chapter 1]

Motivation. Category theory is a useful language to describe sheaves. In mathematics, we often work with objects that have some "nice" structure-preserving maps between them. For example

- topological spaces with continuous maps,
- groups with homomorphisms,
- vector spaces with linear maps.

Category theory studies interactions between the objects of a given "branch of mathematics" just in the language of those maps, without taking into account the structure of the objects themselves. What do all those situations have in common? We have, for instance, a notion of being "essentially the same" for two objects in each of the described examples — a homeomorphism for topological spaces, an isomorphism for groups, etc. What other similarities can we find? What are the differences between the interactions of different types of objects?

2.2.1 Category definition.

We can study many areas of mathematics at the same time by studying classes of objects, and arrows between them. A category is essentially a directed multigraph with an additional structure of composition for the arrows. But the formal definitions get a bit technical, as the theory aims to deal with very general situations.

Definition 2.2.1 (Category)**.** A *category* $\mathcal{C}$ is

(1) a class[1] of *objects* $\mathcal{O}$,

(2) a class of *arrows* (or *morphisms*) $\mathcal{A}$,

(3) functions dom, codom, id, $\circ$:

[1] We can imagine 'set' here. The problem is, that we want to work with categories where the objects are, e.g., all sets — which is itself not a set, but a *proper class* (see, e.g., Russell's paradox or Cantor's paradox). In this course, we will assume to have a set of *allowable* sets, a *universe* which we work with. More general treatment of this problem is a question of set theory and foundations of mathematics. For more comments on this topic, see [Mac98, Ch. 1 §6] or [AHS09, §2].

- $\mathrm{dom} : \mathcal{A} \to \mathcal{O}$, (says where an arrow is going from[2])
- $\mathrm{codom} : \mathcal{A} \to \mathcal{O}$, (says where an arrow is going to)
- $\mathrm{id} : \mathcal{O} \to \mathcal{A}$, (assigns a loop arrow as the identity morphism to an object)
- $\circ : \mathcal{A} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A}$, (the composition rule for the arrows),
  where $\mathcal{A} \times_{\mathcal{O}} \mathcal{A} = \{(g, f) \in \mathcal{A} \times \mathcal{A} \,|\, \mathrm{dom}\, g = \mathrm{codom}\, f\}$ is the set of composable arrows,

such that the following axioms hold:

(a) • $\mathrm{dom}(\mathrm{id}(C)) = C = \mathrm{codom}(\mathrm{id}(C))$, (identity arrow of C is a loop at C: $\begin{tikzcd} C \arrow[loop right, "\mathrm{id}(C)"] \end{tikzcd}$)

• $\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$, $\mathrm{codom}(g \circ f) = \mathrm{codom}\, g$,

$$\left(\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd} \right)$$

for each object C and pair $(g, f) \in \mathcal{A} \times_{\mathcal{O}} \mathcal{A}$,

(b) • $(h \circ g) \circ f = h \circ (g \circ f)$ (associativity)

• $f \circ \mathrm{id}(A) = f$, $\mathrm{id}(B) \circ f = f$ (identity)

for $(f, g), (g, h) \in \mathcal{A} \times_{\mathcal{O}} \mathcal{A}$, $\mathrm{dom}\, f = A$, $\mathrm{codom}\, f = B$.

That is, these diagrams commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g \circ f"] & B \arrow[d, "h \circ g"] \\ C \arrow[r, "h"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\mathrm{id}"] \arrow[dr, "f"'] & & A \arrow[dl, "f"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, "\mathrm{id}"] & & B \end{tikzcd}$$

Some useful notation we will use:

- $\mathrm{Ob}(\mathcal{C}) = \mathcal{O}$, $\mathrm{Mor}(\mathcal{C}) = \mathcal{A}$,
- $\mathrm{Hom}_{\mathcal{C}}(A, B) = \{f \in \mathrm{Mor}(\mathcal{C}) \,|\, \mathrm{dom}\, f = A, \mathrm{codom}\, f = B\}$,[3] (or $\mathrm{Hom}(A, B)$ if $\mathcal{C}$ is clear)
- $\mathrm{id}_A = \mathrm{id}(A)$ or just $\mathrm{id} = \mathrm{id}(A)$ if A is clear from the context.

Examples of categories.

1) Category with just one object and one morphism: $\begin{tikzcd} \bullet \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$

2) Let $(P, \leq)$ be a partially ordered set. We can take a category $\mathcal{P}$ such that $\mathrm{Ob}\, \mathcal{P} = P$, and $\mathrm{Hom}_{\mathcal{P}}(a, b)$ consists of a single arrow if $a \leq b$, and is empty otherwise. The composition rule is clear, as we can always only choose the one arrow between the domain and codomain of the composition.

$$\begin{tikzcd} & \mathrm{d} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{b} \arrow[dr, dash] & & \mathrm{c} \arrow[dl, dash] \\ & \mathrm{a} & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & d \arrow[loop right] & \\ b \arrow[ur] \arrow[loop left] & & c \arrow[ul] \arrow[loop right] \\ & a \arrow[ul] \arrow[ur] \arrow[uu] \arrow[loop right] & \end{tikzcd}$$

We see one simple discrete example in the figure above. Another useful example is to view the real numbers as a poset category — objects are $\mathbb{R}$, and we have one morphism $x \to y$ if $x \leq y$, and no morphisms otherwise.

3) **Set** is the category of sets.

$\mathrm{Ob}(\mathbf{Set})$ are (allowable) sets

[2]Compare to a definition of a quiver. When we have possibly multiple arrows going from one vertex to another, we can not have the set of edges defined just as pairs of vertices as for classical (di)graphs. We could not tell two edges with the same source and target apart. One way around this is to define the quiver as a quadruple — set of vertices, (abstract) set of arrows, and two mappings from arrows to vertices defining the source and the target of each arrow.

[3]In literature, the notation $\mathcal{C}(A, B)$ is also often used for the collection of all morphisms in category $\mathcal{C}$ from object A to object B.

Mor(**Set**) are maps between (allowable) sets

4) **Ab** is the category of abelian groups.

   Ob(**Ab**) are abelian groups

   Mor(**Ab**) are group homomorphisms

5) $\mathbf{Vect}_k$ is the category of vector spaces over a fixed field k.

   Ob($\mathbf{Vect}_k$) are vector spaces

   Mor($\mathbf{Vect}_k$) are linear maps

   We often omit the field k from the notation, and write just **Vect**. This still denotes category of vector spaces over some fixed field k.

6) **Top** is the category of sets.

   Ob(**Top**) are topological spaces

   Mor(**Top**) are continuous maps

7) Suppose that (X, τ) is a topological space. We can define an associated *open set category*, $\mathcal{C} = \mathbf{Open}(X, \tau)$, where

   $\mathrm{Ob}(\mathcal{C}) = \tau$ are the open sets,

   $\mathrm{Mor}(\mathcal{C})$ are inclusion maps, i.e., there is a single arrow $U \longrightarrow V$iff $U \subseteq V$.

   In other words, if we look at the poset $(\tau, \subseteq)$, the open set category is a realisation of this poset as a category, as in example 2).

The terminology of categories is set up so that we naturally interpret objects as sets (possibly with some structure), and morphisms as maps (preserving the structure). However, as examples 1) and 2) show, this is not necessarily the case.

Terminology. We say a category $\mathcal{C}$ is

- *large* if $\mathrm{Ob}(C)$, $\mathrm{Mor}(C)$ are proper classes, i.e. not sets,
- *small* if $\mathrm{Ob}(C)$, $\mathrm{Mor}(C)$ are sets,
- *locally small* (sometimes 'has small Hom-sets') if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for each $A, B \in \mathrm{Ob}(\mathcal{C})$, even if $\mathrm{Ob}(\mathcal{C})$ is a proper class itself.

In the examples above, 3, 4, 5, 6) are all large, but locally small; and 1, 2, 7) are small.

2.2.2 Functors: maps between categories, 'metamorphisms'

A functor is a pair of maps — one for objects, one for morphisms — such that the image of an arrow $A \xrightarrow{f} B$goes between the images of the objects A, B, and the map on the arrows is basically a 'homomorphism' with respect to the composition.

Definition 2.2.2 (Functor)**.** Suppose that $\mathcal{B}$ and $\mathcal{C}$ are categories. A *functor* $T : \mathcal{B} \to \mathcal{C}$ is a pair $(T_{\mathcal{O}}, T_{\mathcal{A}})$, where

$$T_{\mathcal{O}} : \mathrm{Ob}(\mathcal{B}) \longrightarrow \mathrm{Ob}(\mathcal{C}) \text{ and}$$
$$T_{\mathcal{A}} : \mathrm{Mor}(\mathcal{B}) \longrightarrow \mathrm{Mor}(\mathcal{C})$$

are maps which satisfy

- $T_{\mathcal{A}}(\mathrm{id}(B)) = \mathrm{id}(T_{\mathcal{O}}(B))$, (identity goes to identity)

and one of the following two possibilities. Either

- $\mathrm{dom}(T_{\mathcal{A}}(f)) = T_{\mathcal{O}}(\mathrm{dom}(f))$, $\left(A \xrightarrow{f} B \quad \overset{T}{\rightsquigarrow} \quad T_{\mathcal{O}}(A) \xrightarrow{T_{\mathcal{A}}(f)} T_{\mathcal{O}}(B) \right)$

- $\mathrm{codom}(T_{\mathcal{A}}(f)) = T_{\mathcal{O}}(\mathrm{codom}(f))$,[4]
- $T_{\mathcal{A}}(g \circ f) = T_{\mathcal{A}}(g) \circ T_{\mathcal{A}}(f)$, $\left(\begin{tikzcd} T_{\mathcal{O}}(A) \arrow[r, "T_{\mathcal{A}}(f)"] \arrow[rr, bend right, "T_{\mathcal{A}}(g\circ f)"'] & T_{\mathcal{O}}(B) \arrow[r, "T_{\mathcal{A}}(g)"] & T_{\mathcal{O}}(C) \end{tikzcd} \text{ commutes} \right)$

in which case we call it *covariant* functor, or

- $\mathrm{dom}(T_{\mathcal{A}}(f)) = T_{\mathcal{O}}(\mathrm{codom}(f))$, $\left(A \xrightarrow{f} B \quad \overset{T}{\rightsquigarrow} \quad T_{\mathcal{O}}(A) \xleftarrow{T_{\mathcal{A}}(f)} T_{\mathcal{O}}(B) \right)$
- $\mathrm{codom}(T_{\mathcal{A}}(f)) = T_{\mathcal{O}}(\mathrm{dom}(f))$,
- $T_{\mathcal{A}}(g \circ f) = T_{\mathcal{A}}(f) \circ T_{\mathcal{A}}(g)$, $\left(\begin{tikzcd} T_{\mathcal{O}}(A) & T_{\mathcal{O}}(B) \arrow[l, "T_{\mathcal{A}}(f)"'] & T_{\mathcal{O}}(C) \arrow[l, "T_{\mathcal{A}}(g)"'] \arrow[ll, bend left, "T_{\mathcal{A}}(g\circ f)"] \end{tikzcd} \text{ commutes} \right)$

in which case we call it *contravariant*[5] functor.

Examples of functors.

1) The power set of a set S is $\wp(S) = \{R \subseteq S\}$. We can define a functor

$$\begin{aligned} P : \mathbf{Set} &\longrightarrow \mathbf{Set} \\ S &\overset{P_{\mathcal{O}}}{\longmapsto} \wp(S) \\ f &\overset{P_{\mathcal{O}}}{\longmapsto} (R \mapsto f(R)). \end{aligned}$$

2) The *forgetful functor* sending a set with a structure to just the set without any additional structure. For example For : $\mathbf{Ab} \to \mathbf{Set}$ sending a group $(G, +)$ to G, and a homomorphism f between two groups to the same map f between the underlying sets.

3) *Homology* of a given fixed dimension is a covariant functor

$$\begin{aligned} H_n : \mathbf{Top} &\longrightarrow \mathbf{Ab} \\ X &\longmapsto H_n(X) \\ X \xrightarrow{f} Y &\longmapsto H_n(X) \xrightarrow{H_n f} H_n(Y), \end{aligned}$$

where $H_n(X)$ is the n-th singular homology of X with some fixed coefficients, and $H_n f$ is the homomorphism induced by f.

4) *Cohomology* of a given fixed dimension is a contravariant functor

$$\begin{aligned} H^n : \mathbf{Top} &\longrightarrow \mathbf{Ab} \\ X &\longmapsto H^n(X) \\ X \xrightarrow{f} Y &\longmapsto H^n(X) \xleftarrow{H^n f} H^n(Y), \end{aligned}$$

where $H^n(X)$ is the n-th singular cohomology of X with some fixed coefficients, and $H^n f$ is the homomorphism induced by f.

5) Thinking about homology and cohomology, we might also think of the *fundamental group.* There is a slight technicality to discuss. For the definition of fundamental group, we need a topological space together with a fixed point. Therefore, we need to first define a category of pointed topological spaces **pTop**. The objects are topological spaces with a base-point,

[4] We state the conditions on domain and codomain explicitly for clarity of the definition, but they are already consequences of preserving the identity and preserving the composition. For example for the first one, we have $T_{\mathcal{A}}(f) = T_{\mathcal{A}}(f \circ \mathrm{id}_{\mathrm{dom} f}) = T_{\mathcal{A}}(f) \circ T_{\mathcal{A}}(\mathrm{id}_{\mathrm{dom} f}) = T_{\mathcal{A}}(f) \circ \mathrm{id}_{T_{\mathcal{O}}(\mathrm{dom} f)}$, so in order for those to be composable, we need to have $\mathrm{dom}(T_{\mathcal{A}}(f)) = \mathrm{codom}(\mathrm{id}_{T_{\mathcal{O}}(\mathrm{dom} f)}) = T_{\mathcal{O}}(\mathrm{dom}(f))$.

[5] If we say 'functor', we will assume covariance. For more details on covariant vs contravariant, see [Mac98, Ch. 2 §6]

and the morphisms are only those continuous maps that send the base-point of one space to the base point of the other space. Then we can define a covariant functor

$$\begin{aligned} \pi_1 : \mathbf{pTop} &\longrightarrow \mathbf{Grp} \\ (X, x_0) &\longmapsto \pi_1(X, x_0) \\ (X, x_0) \xrightarrow{f} (Y, y_0) &\longmapsto \pi_1(X, x_0) \xrightarrow{\pi_1 f} \pi_1(Y, y_0), \end{aligned}$$

where **Grp** is the category of groups (not necessarily abelian).

6) For a category $\mathcal{C}$, we can define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ by flipping all the arrows:
   - $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) \coloneqq \mathrm{Ob}(\mathcal{C})$,
   - $x \to y \in \mathrm{Mor}(\mathcal{C}^{\mathrm{op}})$ iff $x \leftarrow y \in \mathrm{Mor}(\mathcal{C})$, that is, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) \coloneqq \mathrm{Hom}_{\mathcal{C}}(y, x)$,
   - $f \circ^{\mathrm{op}} g \coloneqq g \circ f$, where $\circ^{\mathrm{op}}$ is the composition in $\mathcal{C}^{\mathrm{op}}$, and $\circ$ is the composition in $\mathcal{C}$.

   Assume that $F : \mathcal{B} \to \mathcal{C}$ is a contravariant functor. Then F defines a covariant functor $F : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ (or also $F : \mathcal{B} \to \mathcal{C}^{\mathrm{op}}$).

7) The *duality* in vector spaces is a contravariant functor. For **Vect**, the category of vector spaces over a fixed field k, it is defined as follows:

$$\begin{aligned} D : \mathbf{Vect} &\longrightarrow \mathbf{Vect} \\ V &\longmapsto V^* = \{\text{linear functions from } V \text{ to } k\} \\ V \xrightarrow{\gamma} W &\longmapsto V^* \xleftarrow{\gamma^*} W^*, \text{ given by } \gamma^*(f) = f \circ \gamma \end{aligned}$$

8) Important examples of functors are the Hom-*functors*. Let $\mathcal{C}$ be a category, and X be an object $\mathcal{C}$. We define a covariant functor

$$\begin{aligned} \mathrm{Hom}(X, -) : \mathcal{C} &\longrightarrow \mathbf{Set} \\ V &\longmapsto \mathrm{Hom}(X, V) \\ V \xrightarrow{f} W &\longmapsto \mathrm{Hom}(X, V) \xrightarrow{f_*} \mathrm{Hom}(X, W), \text{ given by } f_*(\varphi) = f \circ \varphi, \end{aligned}$$

   and a contravariant functor

$$\begin{aligned} \mathrm{Hom}(-, X) : \mathcal{C} &\longrightarrow \mathbf{Set} \\ V &\longmapsto \mathrm{Hom}(V, X) \\ V \xrightarrow{f} W &\longmapsto \mathrm{Hom}(V, X) \xleftarrow{f^*} \mathrm{Hom}(W, X), \text{ given by } f^*(\varphi) = \varphi \circ f. \end{aligned}$$

   If a functor F from $\mathcal{C}$ to **Set** can be 'expressed' as a Hom-functor, it is called *representable* (discussed more in Lecture 3, Yoneda lemma).

   Note that the example 7) is a special case of contravariant hom-functor, $\mathrm{Hom}_{\mathbf{Vect}}(-, k)$, where k is the (arithmetic) one-dimensional vector space.

9) *Representations of quivers* can be viewed as functors. For a quiver Q, we can consider $\mathcal{Q}$ a category 'generated' by Q. The objects are vertices of Q, and the morphisms $\mathrm{Hom}(u, v)$ are all paths between u and v in Q — including the empty path for $\mathrm{Hom}(u, u)$, which is the identity of u. The composition is the composition of paths.

   A representation of Q is a functor to the category of vector spaces, assigning a vector space to each vertex, and a linear mapping to each arrow. For example, if

$$Q = \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots\,,$$

   a representation of $\mathcal{Q}$ is

$$\begin{aligned} R : \mathcal{Q} &\longrightarrow \mathbf{Vect} \\ v_i &\longmapsto V_i \\ v_i \to v_{i+1} &\longmapsto M_i : V_i \to V_{i+1}, \end{aligned}$$

   which can be, for example, a persistence module of a simplicial complex with an ordering. Note that the way we set $\mathcal{Q}$ up, we can assign an arbitrary linear map to each arrow of Q independently.

Lecture 3: Natural transformation, equivalence, adjoint pairs

References. [Mac98, Ch. 2 §4, Ch. 3, Ch. 4 §1] [Cur14, Ch. 2 §4, Ch. 3, Ch. 4 §1]

Natural transformations: morphisms between functors, 'metametamorphisms'

It is useful to have a way to compare functors to each other. We can see natural transformations between functors as an analogy to homotopy between continuous maps. Having defined functors, we might think about some notion of equivalence of categories. It is not difficult to see that functors can be composed, and we can also define an identity functor, where both maps act identically. Therefore, we could consider two functors going in opposite directions between two categories, such that their composition is the identity functor. However, we can immediately see that his notion of equivalence is problematic in several ways. First, it is a bit fishy to talk about invertible maps between something that can be a class; second, we should not really care if we throw away some (or all) "isomorphic copies" of an object from a category, which would cause problems for this notion. A solution to this is very similar to what we do in topology. The notion of homeomorphism can be too strict to capture the kind of "sameness" we are looking for, and so we introduce the notion of homotopy — a "map between maps". And with homotopy at hand, we can define a notion of equivalence for the spaces too. We do not need the composition of continuous maps between the spaces to be equal to identity, it is enough for it to be homotopic to the identity.

Definition 2.3.3 (Natural transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be two (covariant) functors. A *natural transformation* τ from F to G is a collection of morphisms $F(A) \xrightarrow{\tau(A)} G(A)$ in $\mathrm{Mor}(\mathcal{D})$, one for each object $A \in \mathrm{Ob}(\mathcal{C})$, such that for every $A \xrightarrow{f} B$ in $\mathrm{Mor}(\mathcal{C})$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\tau(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau(B)"] & G(B) \end{tikzcd},$$

i.e., $\tau(B) \circ F(f) = G(f) \circ \tau(A)$.

Example of a natural transformation.

Let $R, S : Q \to \mathbf{Vect}$ be two representations of a quiver Q: $v_1 \longrightarrow v_2 \longrightarrow v_3$. A natural transformation $\tau : R \to S$ is a collection of maps between the functors' images such that the following diagram commutes:

$$\begin{tikzcd} R(v_1) \arrow[r, "\tau(v_1)"] \arrow[d] & S(v_1) \arrow[d] \\ R(v_2) \arrow[r, "\tau(v_2)"] \arrow[d] & S(v_2) \arrow[d] \\ R(v_3) \arrow[r, "\tau(v_3)"] & S(v_3) \end{tikzcd}.$$

Definition 2.3.4. For any locally small category $\mathcal{C}$, each object $c \in \mathcal{C}$ defines a functor $\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathbf{Set}$. Functors of this form are called *representable* functors.

Definition 2.3.5. Let $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ be the category of functors defined by:

$$\mathrm{Ob}(\mathrm{Fun}(\mathcal{C}, \mathbf{Set})) \coloneqq \{\text{functors } F : \mathcal{C} \to \mathbf{Set}\}$$
$$\mathrm{Mor}(\mathrm{Fun}(\mathcal{C}, \mathbf{Set})) \coloneqq \{\text{natural transformations of the functors}\}.$$

This also works for general categories $\mathcal{C}, \mathcal{D}$. Now we may wish to relate functors to those which are representable.

Lemma 2.3.6 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a locally small category and each $F \in \mathrm{Ob}(\mathrm{Fun}(\mathcal{C}, \mathbf{Set}))$, there is a (natural) bijection* [6]

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(c,-),F) \simeq F(c)$$

Remark. Natural bijection here means the squares induced by morphisms $c \xrightarrow{f} c'$ in $\mathcal{C}$ commute.

Corollary 2.3.7. *Let c, c' be elements in a locally small category $\mathcal{C}$. Then we have the following natural bijection.*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}_{\mathcal{C}}(c,-),\mathrm{Hom}_{\mathcal{C}}(c',-)) \simeq \mathrm{Hom}_{\mathcal{C}}(c',c)$$

Therefore we can think of $\mathcal{C}$ as being embedded in $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$. This corollary, in a sense, suggests all objects are determined by their relations to other objects.

Example

1. Let $\mathcal{C} = \mathbf{Open}(X)$ be a category of open sets with inclusion forming the morphisms. This corollary implies we lose no information about the topology of C by simply considering the functors on the topology.

2. Consider $\mathbb{Z}$ as a category. We can determine a secret object x by playing a game which asks questions of the form "Is there a morphism $x \to 10$?". If the answer is "yes" then we know $x \leq 10$. After asking sufficiently many questions we can determine x, illustrating how objects are determined by morphisms.

Proposition 2.3.8. *Suppose $S : \mathcal{B} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{D}$ are functors. Then*

$$\begin{aligned} (T \circ S)(b) &\coloneqq T(S(b)) \text{ for } b \in \mathrm{Ob}(\mathcal{B}) \\ (T \circ S)(b \to b') &\coloneqq T(S(b \to b')) \text{ for } b \to b' \in \mathrm{Mor}(\mathcal{B}) \end{aligned}$$

defines a functor from $\mathcal{B}$ to $\mathcal{D}$.

Proof. First check $(T \circ S)(\mathrm{id}_B) = \mathrm{id}_{(T\circ S)(B)}$.

$$\begin{aligned} T(S(\mathrm{id}_B)) &= T(\mathrm{id}_{S(B)}) \\ &= \mathrm{id}_{T(S(B))} \\ &= \mathrm{id}_{(T\circ S)(B)} \end{aligned}$$

Now check the composition rule. Let $f : a \to b$ and $g : b \to c$,

$$\begin{aligned} (T \circ S)(g \circ f) &= T(S(g) \circ S(f)) \\ &= T(S(g)) \circ T(S(f)) \\ &= (T \circ S)(g) \circ (T \circ S)(f) \end{aligned}$$

So we conclude $T \circ S : \mathcal{B} \to \mathcal{D}$ is a functor. □

Definition 2.3.9.

- $f \in \mathrm{Mor}(\mathcal{C})$ is called an *isomorphism* if there exists $g \in \mathrm{Mor}(\mathcal{C})$ such that $f \circ g = \mathrm{id}(\mathrm{dom}\, g)$ and $g \circ f = \mathrm{id}(\mathrm{dom}\, f)$.

- Two objects $a, b \in \mathrm{Ob}(\mathcal{C})$ are called *isomorphic* if there exists an isomorphism $f : a \to b$.

- Two categories $\mathcal{B}, \mathcal{C}$ are *isomorphic* if there exists functors $F, G : \mathcal{B} \to \mathcal{C}$, such that $F \circ G = \mathrm{Id}_{\mathcal{C}}$ and $G \circ F = \mathrm{Id}_{\mathcal{B}}$. This definition of isomorphism is too rigid to be useful in distinguishing categories. For this we introduce the following notion.

- A natural transformation $\tau : F \to G$ that is also an isomorphism (i.e. it has an inverse natural transformation) in the category $\mathrm{Fun}(\mathcal{B}, \mathcal{C})$ is called a *natural isomorphism*.

[6] Note that some sources instead use the notation $\mathrm{Hom}(\mathcal{B}, \mathcal{C})$ for the category of functors between two categories.

Definition 2.3.10. Let $\mathcal{B}, \mathcal{C}$ be two locally small categories and let $F : \mathcal{B} \to \mathcal{C}$ be a functor. If for all objects $a, b \in \mathcal{B}$ the map $\mathrm{Hom}_{\mathcal{B}}(a,b) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(Fa, Fb)$

- is injective F is called *full*;
- is surjective F is called *faithful*;
- is bijective F is called *fully faithful.*

Example Let $\mathcal{C}$ be a locally small category. The functor

$$\begin{aligned} H : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}, \mathbf{Set}) \\ c &\mapsto \mathrm{Hom}_{\mathcal{C}}(c, -), \end{aligned}$$

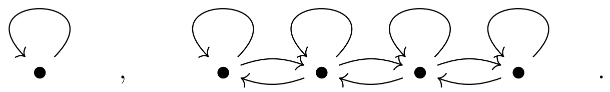
is fully faithful by Corollary 2.3.7. This demonstrates in what sense objects are determined, up to isomorphism, by the morphisms incident to them.

Definition 2.3.11 (Equivalence of Categories, two versions)**.** Let $\mathcal{B}, \mathcal{C}$ be categories and let $F : \mathcal{B} \to \mathcal{C}$ be a functor.

1. We say that $\mathcal{B}$ and $\mathcal{C}$ are *equivalent* categories if there also exists a functor $G : \mathcal{C} \to \mathcal{B}$, and two natural isomorphisms $\sigma : \mathrm{Id}_{\mathcal{C}} \to F \circ G$ and $\tau : G \circ F \to \mathrm{Id}_{\mathcal{B}}$; that is, $F \circ G$ is isomorphic to $\mathrm{Id}_{\mathcal{C}}$ in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, and $G \circ F$ is isomorphic to $\mathrm{Id}_{\mathcal{B}}$ in the category $\mathrm{Fun}(\mathcal{B}, \mathcal{B})$.
2. We say that F is an *equivalence* of categories if F is fully faithful and F is surjective onto the set of isomorphism classes of objects; that is, for each object $c \in \mathrm{Ob}(\mathcal{C})$ there exists $b \in \mathrm{Ob}(\mathcal{B})$ and an isomorphism $Fb \xrightarrow{\sim} c$.

Exercise 2.3.12. *Show that the two versions of the definition above are equivalent.*

Example Let $\mathcal{B}, \mathcal{C}$ be the following two categories



There are four possible functors $\mathcal{B} \xrightarrow{F} \mathcal{C}$ corresponding to the four objects in $\mathcal{C}$ and a single functor $\mathcal{C} \xrightarrow{G} \mathcal{B}$. Take the functor F mapping $\mathcal{B}$ to the first object in $\mathcal{C}$. The horizontal morphisms and their inverses give natural isomorphisms between the functors $F \circ G$ and $\mathrm{Id}_{\mathcal{C}}$, together with $F \circ G = \mathrm{Id}_{\mathcal{B}}$ this implies that the categories are equivalent.

2.3.1 Adjoint Pairs

An overarching goal in the study of category theory is translating calculations from difficult categories to simple ones. We will introduce two equivalent definitions of this notion (note the equivalence is non-obvious).

Definition 2.3.13 (1)**.** Let $\mathcal{B}, \mathcal{C}$ be categories and $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{B}$ be functors between them. We say, (F, G) form an *adjoint pair*, or equivalently F is a *left adjoint* of G, or G is *right adjoint* of F, if there exist natural transformations $\sigma : \mathrm{Id}_{\mathcal{C}} \to F \circ G$ and $\tau : G \circ F \to \mathrm{Id}_{\mathcal{B}}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathrm{Id}_{\mathcal{C}} \circ F \arrow[r, "\sigma \circ \mathrm{id}_F"] \arrow[dr, "\mathrm{id}_F"'] & F \circ G \circ F \arrow[d, "\mathrm{id}_F \circ \tau"] \\ & F \circ \mathrm{Id}_{\mathcal{B}}, \end{tikzcd} \qquad \begin{tikzcd} G \circ \mathrm{Id}_{\mathcal{C}} \arrow[r, "\mathrm{id}_G \circ \sigma"] \arrow[dr, "\mathrm{id}_G"'] & G \circ F \circ G \arrow[d, "\tau \circ \mathrm{id}_G"] \\ & \mathrm{Id}_{\mathcal{B}} \circ G \end{tikzcd} .$$

(See additional notes 6.1.1 on Functor compositions)

Definition 2.3.14 (2)**.** Let $\mathcal{B},\mathcal{C}$ be categories and $F:\mathcal{B}\to\mathcal{C}$, $G:\mathcal{C}\to\mathcal{B}$ be functors between them. (F,G) form an adjoint pair if there exist bijections $\phi_{b,c}:\mathrm{Hom}_{\mathcal{C}}(Fb,c)\to\mathrm{Hom}_{\mathcal{B}}(b,Gc)$ for each pair $b\in\mathrm{Ob}(\mathcal{B}), c\in\mathrm{Ob}(\mathcal{C})$, such that for all $f:b\to b'\in\mathrm{Mor}(\mathcal{B})$ and $g:c\to c'\in\mathrm{Mor}(\mathcal{C})$ the following diagrams commute,

$$\begin{tikzcd}
\phi_{b,c}:\mathrm{Hom}_{\mathcal{C}}(Fb,c) \arrow[r, "\sim"] \arrow[d, "g\circ-"] & \mathrm{Hom}_{\mathcal{B}}(b,Gc) \arrow[d, "G(g)\circ-"] \\
\phi_{b,c'}:\mathrm{Hom}_{\mathcal{C}}(Fb,c') \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{B}}(b,Gc'),
\end{tikzcd}$$

$$\begin{tikzcd}
\phi_{b,c}:\mathrm{Hom}_{\mathcal{C}}(Fb,c) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{B}}(b,Gc) \\
\phi_{b',c}:\mathrm{Hom}_{\mathcal{C}}(Fb',c) \arrow[r, "\sim"] \arrow[u, "-\circ F(f)"] & \mathrm{Hom}_{\mathcal{B}}(b',Gc). \arrow[u, "-\circ f"]
\end{tikzcd}$$

Example The discrete topology functor and the forgetful functor form an adjoint pair $(\mathcal{D},\mathbf{For})$.

$$\begin{aligned}
\mathbf{For}:\mathbf{Top}&\to\mathbf{Set}\\
\mathcal{D}:\mathbf{Set}&\to\mathbf{Top}\\
X&\mapsto(X,\text{discrete topology})
\end{aligned}$$

Consider $X\in\mathrm{Ob}(\mathbf{Set})$ and $Y\in\mathrm{Ob}(\mathbf{Top})$.

$$\mathrm{Hom}_{\mathbf{Top}}(\mathcal{D}X,Y)\simeq\mathrm{Hom}_{\mathbf{Set}}(X,\mathbf{For}Y).$$

Alternatively we could use the map sending the set X to the topological space X with the trivial topology.

$$\begin{aligned}
\mathbf{For}:\mathbf{Top}&\to\mathbf{Set}\\
\mathcal{E}:\mathbf{Set}&\to\mathbf{Top}\\
X&\mapsto(X,\text{trivial topology})
\end{aligned}$$

Consider $X\in\mathrm{Ob}(\mathbf{Set})$ and $Y\in\mathrm{Ob}(\mathbf{Top})$.

$$\mathrm{Hom}_{\mathbf{Top}}(\mathcal{E}X,Y)\simeq\mathrm{Hom}_{\mathbf{Set}}(X,\mathbf{For}Y).$$

Similarly we can form an adjoint pair between **Vect** and **Set** with the functors:

$$\begin{aligned}
\mathbf{For}:\mathbf{Vect}&\to\mathbf{Set}\\
\mathcal{F}:\mathbf{Set}&\to\mathbf{Vect}\\
B&\mapsto\mathcal{F}B,
\end{aligned}$$

where $\mathcal{F}B$ is the real vector space with basis B. Consider $B\in\mathrm{Ob}(\mathbf{Set})$ and $V\in\mathrm{Ob}(\mathbf{Vect})$.

$$\mathrm{Hom}_{\mathbf{Vect}}(\mathcal{F}B,V)\simeq\mathrm{Hom}_{\mathbf{Set}}(B,\mathbf{For}V).$$

So following definition (2), in all instances, $(\mathcal{D},\mathbf{For}),(\mathcal{E},\mathbf{For}),(\mathcal{F},\mathbf{For})$ form adjoint pairs.

Lecture 4 & 5: Coproducts, Colimits, Products and Limits

2.4.1 Coproducts

Motivating Example

Let X be a topological space. The set of open sets in X is a poset, with $U\le V$ if and only if $U\subseteq V$. We denote by $\mathcal{C}=\mathbf{Open}(X)$ the associated category and we note that $\mathcal{C}=\mathbf{Open}(X)$ has extra structure. Namely, if $U,V\in\mathrm{Ob}(\mathbf{Open}(X))$, then $U\cup V\in\mathrm{Ob}(\mathbf{Open}(X))$.

Question Do other categories have a similar structure?

Observations

1. $U \to U \cup V$,
2. $V \to U \cup V$,
3. If $U \subseteq W$ and $V \subseteq W$, then $U \cup V \subseteq W$.

In other words,

1. $\exists\ U \to U \cup V$,
2. $\exists\ V \to U \cup V$,
3. If $\exists\ U \subseteq W$ and $\exists\ V \subseteq W$, then $\exists\ U \cup V \subseteq W$ such that

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & U \cup V \arrow[d, dashed, "\exists!"] & V \arrow[l] \arrow[dl] \\ & W & \end{tikzcd}$$

commutes.

Exercise Let $\mathcal{C} = \mathbf{Vect}$. Let $V, W \in \mathrm{Ob}(\mathbf{Vect})$. Find $X \in \mathrm{Ob}(\mathbf{Vect})$ such that

1. $\exists\ V \to X$,
2. $\exists\ W \to X$,
3. If $\exists\ Y \to Y$ and $\exists\ W \to Y$, then $\exists!\ X \to Y$ such that

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & X \arrow[d, dashed, "\exists!"] & W \arrow[l] \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes.

Solution Take $X = V \times W = \{(v, w) : v \in V, w \in W\}$. In particular, the map $V \to X$ sends v to $(v, 0)$, the map $W \to X$ sends w to $(0, w)$ and if $\phi_v : V \to Y$, $\phi_w : W \to Y$, then the map $X = V \times W \to Y$ sends (v, w) to $\phi_v(v) + \phi_w(w)$.

Definition 2.4.15 (Coproduct)**.** A *coproduct* of $a, b \in \mathrm{Ob}(\mathcal{C})$ is

1. an object $a \sqcup b \in \mathrm{Ob}(\mathcal{C})$,
2. maps $a \to a \sqcup b$, $b \to a \sqcup b$ such that if $a \to c \in \mathrm{Mor}(\mathcal{C})$ and $b \to c \in \mathrm{Mor}(\mathcal{C})$, then $\exists! a \sqcup b \to c \in \mathrm{Mor}(\mathcal{C})$ such that

$$\begin{tikzcd} a \arrow[r] \arrow[dr] & a \sqcup b \arrow[d, dashed, "\exists!"] & b \arrow[l] \arrow[dl] \\ & c & \end{tikzcd}$$

commutes.

Remark. Coproducts may not exist.

Exercise If $a \sqcup b$ exists, then it is unique (up to isomorphism).

Coproducts over families

Motivation If $\{U_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathbf{Open}(X))$, then $\bigcup_{i\in I} U_i \in \mathrm{Ob}(\mathbf{Open}(X))$.

Definition 2.4.16 (Coproduct over a family of objects)**.** If $\{c_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$, then the *coproduct* $\bigsqcup_{i\in I} c_i$ is

1. an object in $\mathcal{C}$,
2. maps $c_j \to \bigsqcup_{i\in I} c_i$ for each $j \in I$ such that if there is an object c' and maps $c_j \to c'$ for each $j \in I$ then $\exists!$ map $\bigsqcup_{i\in I} c_i \to c'$ such that

$$\begin{tikzcd}
\bigsqcup_{i\in I} c_i \arrow[d, dashed, "\exists!"] & c_j \arrow[l, "*"'] \arrow[dl, "*"] \\
c' &
\end{tikzcd}$$

commutes for all $j \in I$.

Remark. The maps denoted by $*$ depend on j, while the vertical map is independent of j.

Example 2.4.17. 1. If $\{U_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathbf{Open}(X))$, then $\bigsqcup_{i\in I} U_i = \bigcup_{i\in I} U_i \in \mathrm{Ob}(\mathbf{Open}(X))$.

2. Let $I = \mathbb{N}$, $V_i = k \in \mathrm{Ob}(\mathbf{Vect})$. Take $Z = k \in \mathrm{Ob}(\mathbf{Vect})$ and $\phi_i : V_i \to Z$ is the identity map. Then $\bigsqcup_{i\in I} V_i = \bigoplus_{i\in I} V_i \coloneqq \{(v_1, v_2, \cdots) : v_i = 0 \text{ for all but finitely many } i \in I\}$.

Remark. One could have first guessed that $\prod_{i\in I} V_i \coloneqq \{(v_1, v_2, \cdots) : v_i \in V_i\}$ is the coproduct in the above. However, this is not the case as the vertical dashed map below

$$\begin{tikzcd}
\prod_{i\in I} V_i \arrow[d, dashed] & V_j \arrow[l] \arrow[dl, "id"] \\
Z &
\end{tikzcd}$$

given by mapping $(v_1, v_2, \cdots)$ to $\sum_{i\in I} v_i$ may not converge.

Corollary 2.4.18. *$\prod_{i\in I} V_i = \bigoplus_{i\in I} V_i$ if I is finite.*

2.4.2 Colimits

One way to *pick out* specific objects in a category C, is to use what is called an *indexing category.* For example, let I be the following discrete category:

$$\begin{tikzcd}
\bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \cdots .
\end{tikzcd}$$

and let $F : I \to C$ be a functor, which picks an object i in the category I and sends it to the object c_i in the category C. Then $\bigsqcup_{i\in I} c_i = \bigsqcup_{i\in I} F(i)$. But what if I had more arrows?

Example 2.4.19. Let I be the discrete category generated by the following diagram (in particular, all compositions of arrows should also be arrows in I):

$$\begin{tikzcd}
1 \arrow[loop above] \arrow[r] & 2 \arrow[loop above] \arrow[r] & 3 \arrow[loop above]
\end{tikzcd}$$

Let $F : I \to \mathbf{Open}(X)$. Then $\bigsqcup_I F(i) = \bigcup_I F(i) = F(3)$, because $F(1) \subseteq F(2) \subseteq F(3)$.

Definition 2.4.20 (Colimit of a functor)**.** Let I be a locally small category and $F : I \to \mathcal{C}$ be a functor. The *colimit* of F, denoted $\varinjlim F$, is

1. an object $\varinjlim F \in \mathcal{C}$,
2. maps $F(i) \to \varinjlim F$ for each $i \in \mathrm{Ob}(I)$ such that:
   - If $i \to j \in \mathrm{Mor}(I)$, then
     $$\begin{tikzcd} F(i) \arrow[r] \arrow[d] & F(j) \arrow[dl] \\ \varinjlim F & \end{tikzcd}$$
     commutes.
   - If $F(i) \to c \in \mathrm{Mor}(\mathcal{C})$ for each $i \in I$ then there exists a unique map $\varinjlim F \to c$ such that:
     $$\begin{tikzcd} F(i) \arrow[rr] \arrow[dr, "given"] \arrow[ddr, "if"'] & & F(j) \arrow[dl, "given"'] \arrow[ddl, "if"] \\ & \varinjlim F \arrow[d, dashed, "\exists!"] & \\ & c & \end{tikzcd}$$
     commutes whenever $i \to j \in \mathrm{Mor}(I)$.

Example 2.4.21. Let I be the following category.

$$I = \begin{tikzcd} \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \end{tikzcd}$$

and let $F : I \to \mathbf{Open}(X)$. Then $\varinjlim F = F(3)$.

Example 2.4.22. Let I be the category of negative integers with a morphisms from a to b if a is smaller than b. Let $F : I \to \mathbf{Vect}$ be a functor. Then $\varinjlim F = F(-1)$.

Example 2.4.23. Let $I = \mathbb{N}$ viewed as a poset category and let $F : I \to \mathbf{Vect}$ be a functor:

$$F(I) = V_1 \xrightarrow{\phi_{12}} V_2 \xrightarrow{\phi_{23}} V_3 \longrightarrow \cdots$$

Then $\varinjlim F = \bigoplus_I V_i /_\sim$, where $(\underbrace{0, \cdots, v}_{\text{j}}, 0, \cdots) \in \bigoplus V_i$ is equivalent to $(\underbrace{0, \cdots, w}_{\text{k}}, 0, \cdots) \in \bigoplus V_i$ if there exists $l \in I$ such that

$$\phi_{l-1,l} \circ \cdots \circ \phi_{j+1,j+2} \circ \phi_{j,j+1}(v) = \phi_{l-1,l} \circ \cdots \circ \phi_{k,k+1}(w).$$

More concretely, let

$$V_i = \begin{cases} \mathbb{R} & \text{if } i \text{ is even} \\ \mathbb{R} \oplus \mathbb{R} & \text{if } i \text{ is odd} \end{cases}$$

where $x \in \mathbb{R}$ is mapped to $(x, 0) \in \mathbb{R} \oplus \mathbb{R}$ and $(x, y) \in \mathbb{R} \oplus \mathbb{R}$ is mapped to $(x, 0) \in \mathbb{R}$. Note that $\bigoplus V_i$ is infinite dimensional, however $\varinjlim F = \mathbb{R}$. This is because $(x, y) \sim (x, z)$ as they eventually both map to $x \in \mathbb{R}$.

Example 2.4.24. Let I be the following category.

$$\begin{tikzcd} & c \arrow[d] \\ a \arrow[r] & b \end{tikzcd}$$

Let $F : I \to \mathbf{Vect}$. Then $\varinjlim F = F(b)$.

Example 2.4.25. Let I be the following category:

$$I = \begin{cases} \mathrm{Ob}(I) = \bullet \\ \mathrm{Mor}(I) = \mathbb{Z} \quad \text{with } a \circ b = a + b \end{cases}$$

Let $F : I \to \mathbf{Vect}$ be a functor defined on the object by $F(\bullet) = V$ and on the morphisms by $F(0) = \mathrm{id} : V \to V$, $F(1) = M : V \to V$. Clearly, $F(n)$ consists on composing M n-times.

Question What is $\varinjlim F$?

Answer Consider the following diagram

$$\begin{tikzcd}
V \arrow[rr, "M"] \arrow[dr, "\phi"] \arrow[ddr, "\psi"'] & & V \arrow[dl, "\phi"'] \arrow[ddl, "\psi"] \\
& \varinjlim F \arrow[d, dashed, "\lambda"] & \\
& W &
\end{tikzcd}$$

Chasing the arrows we get the following relationship $\phi \circ M = \phi$, which implies that $\phi(Mv) - \phi(v) = 0$ and finally that $\phi(Mv - v) = 0$. Similarly, we get that $\psi(Mv - v) = 0$. Let $V_n = \{w \in W : w = Mv - v \text{ for some } v \in V\}$. Suppose $W = V \big/ V_n$, with $\psi : V \to V \big/ V_n$ the quotient map. Consider the following subdiagram of the previous diagram:

$$\begin{tikzcd}
V \arrow[dr, "\phi"] \arrow[ddr, "\psi"'] & \\
& \varinjlim F \arrow[d, dashed, "\lambda"] \\
& V \big/ V_n
\end{tikzcd}$$

Note that since the restriction of ϕ from V to V_n is zero (i.e. $\phi(V_n) = 0$), the map ϕ must factor through $V \big/ V_n$. By commutativity of the diagram and the surjectivity of ψ, there exist a map $\phi' : V \big/ V_n \to \varinjlim F$, such that $\lambda \circ \phi'(w) = w$, which implies that the $\varinjlim F \simeq V \big/ V_n$. In different words, $\varinjlim F$ is the set of M-coinvariants of V and ϕ is the quotient map.

Example 2.4.26. Let I be the following category.

$$\begin{tikzcd}
\bullet \arrow[dr] & & \bullet \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

Let $F : I \to \mathbf{Open}(X)$ be a functor. By applying the functor F, the diagram becomes

$$\begin{tikzcd}
U \arrow[dr] & & V \arrow[dl] \\
& W &
\end{tikzcd}$$

As usual, let's consider the limit diagram in the specific setting

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[ddr] & & U \arrow[dl] \arrow[ddl] \\
& \varinjlim F \arrow[d, dashed] & \\
& Z &
\end{tikzcd}$$

Note that $U \subseteq \varinjlim F$ and if $U \subseteq Z$, then $U \subseteq \varinjlim F \subseteq Z$. The same reasoning applies for V. This implies that $\varinjlim F = U \cup V$, as $U \to U \cup V$, $V \to U \cup V$ and if $U \to Z$ and $V \to Z$ then the following diagram commutes.

$$\begin{tikzcd}
U \arrow[dr] \arrow[ddr] & W \arrow[l] \arrow[r] \arrow[d] & V \arrow[dl] \arrow[ddl] \\
& U \cup V \arrow[d, dashed] & \\
& Z &
\end{tikzcd}$$

Exercise 2.4.27. *Let I be the following category.*

$$I = \begin{tikzcd}
\bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \cdots
\end{tikzcd}.$$

*Let $F : I \to$ **Vect** be a functor with $F(i) = \mathbb{V}_i$ for all i. Show that both the coproduct and the colimit of F are isomorphic to $\bigoplus V_i$.*

Example 2.4.28. Let I be the following category:

$$I = \begin{cases} \mathrm{Ob}(I) = \bullet \\ \mathrm{Mor}(I) = \mathbb{Z} \quad \text{with } a \circ b = a + b \end{cases}$$

Let $F : I \to \mathbf{Vect}$ be a functor defined on the object by $F(\bullet) = V$ and on the morphisms by $F(0) = \mathrm{id} : V \to V$, $F(1) = M : V \to V$. What is $\varinjlim F$?

As per usual, $\varinjlim F$ is an object in **Vect**, such that the following diagram commutes for all $n \in \mathbb{Z}$.

$$\begin{tikzcd}
V \arrow[rr, "F(n)"] \arrow[dr, "\phi"'] & & V \arrow[dl, "\phi"] \\
& \varinjlim F &
\end{tikzcd}$$

If there exists $Z \in \mathrm{Ob}(\mathbf{Vect})$, with $V \to Z$, then the consider the following diagram.

$$\begin{tikzcd}
V \arrow[rr, "F(n)"] \arrow[dr, "\phi"] \arrow[ddr, "\psi"'] & & U \arrow[dl, "\phi"'] \arrow[ddl, "\psi"] \\
& \varinjlim F \arrow[d, dashed] & \\
& Z &
\end{tikzcd}$$

By chasing the arrows in the outermost triangle, we get $\psi(v) = \psi(nv)$ for all $n \in \mathbb{Z}$, which implies that $\psi(v) = \psi(w)$ whenever there exist $n \in \mathbb{Z}$ such that $M^n v = w$.

Let's define an equivalence relation $v \sim w$, if there exist $n \in \mathbb{Z}$ such that $M^n v = w$ and a map $\psi' : V/\sim \to Z$ such that the following diagram commutes.

$$\begin{tikzcd}
V \arrow[r, "q"] \arrow[d, "\psi"'] & V/\sim \arrow[dl, "\psi'"] \\
Z &
\end{tikzcd}$$

Guess: $\varinjlim F = V/\sim$

Consider the following diagram.

$$\begin{tikzcd}
V \arrow[rr, "F(n)"] \arrow[dr, "\phi"] \arrow[ddr, "q"'] & & V \arrow[dl, "\phi"'] \arrow[ddl, "q"] \\
& \varinjlim F \arrow[d, dashed, "\lambda"] & \\
& V/\sim &
\end{tikzcd}$$

By the same argument as above, there exists a ϕ' such that $\phi' = \lambda^{-1}$, which implies that $\varinjlim F = V/\sim$. Lastly, suppose $V_n = \{w \in V : w = MV - v \text{ for some } v \in V\} = \operatorname{im}(M - \operatorname{id}$. Suppose $v \sim w$. Then

$$v + (Mv - v) + (M^2 v - Mv) + \cdots + (M^n - M^{n-1} v) = M^n v = w,$$

which implies that $w \in v + V_n$. Hence $V/\sim \simeq V\big/V_n$, the set of orbits of $\mathbb{Z}$ on V.

2.5.1 Products

Motivating example

Let $U, V \in \mathrm{Ob}(\mathbf{Open}(X))$ and let $U \cap V \in \mathrm{Ob}(\mathbf{Open}(X))$. If $W \subseteq U$ and $W \subseteq V$, then $W \subseteq U \cap V$. The diagrammatic version of the previous statement is the following

$$\begin{tikzcd}
& W \arrow[dl, "if"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "if"] & \\
U & U \cap V \arrow[l, "given"] \arrow[r, "given"'] & V
\end{tikzcd}$$

Question What is an analogous structure in other categories?

Example 2.5.29. Let $X, Y \in \mathrm{Ob}(\mathbf{Vect}(X))$. If $W \in \mathrm{Ob}(\mathbf{Vect}(X))$ maps into X and Y, then

$$\begin{tikzcd}
& W \arrow[dl, "\theta_x"'] \arrow[d, dashed] \arrow[dr, "\theta_y"] & \\
X & X \times Y \arrow[l, "proj."] \arrow[r, "proj."'] & Y
\end{tikzcd}$$

where $W \dashrightarrow X \times Y$ is given by mapping $w \in W$ to $(\theta_x(w), \theta_y(w)) \in X \times Y$.

Definition 2.5.30 (Direct product)**.** The *direct product* of $a, b \in \mathrm{Ob}(\mathcal{C})$, denoted $a \sqcap b$, is

1. an object $a \sqcap b \in \mathrm{Ob}(\mathcal{C})$,
2. maps $a \leftarrow a \sqcap b$, $b \leftarrow a \sqcap b$ such that if $a \leftarrow c \in \mathrm{Mor}(\mathcal{C})$ and $b \leftarrow c \in \mathrm{Mor}(\mathcal{C})$, then $\exists! c \leftarrow a \sqcap b \in \mathrm{Mor}(\mathcal{C})$ such that

$$\begin{tikzcd}
& c & \\
a \arrow[ur] & a \sqcap b \arrow[l] \arrow[r] \arrow[u, dashed, "\exists!"] & b \arrow[ul]
\end{tikzcd}$$

   commutes.

Direct product over families

Motivation Let $\{U_i\}_{i \in I} \subseteq \mathrm{Ob}(\mathbf{Open}(X))$. If $W \in \{U_i\}$ for each $i \in I$, then $W \in \bigcap_{i \in I} U_i$. In other words, the following diagram

$$\begin{tikzcd}
W \arrow[d, dashed] \arrow[dr] & \\
\prod_{i \in I} U_i \arrow[r] & U_j
\end{tikzcd}$$

commutes. Note that if $|I| = \infty$, then $\bigcap_{i \in I} U_i$ might not be open. Hence, infinite direct products might not exist in $\mathbf{Open}(X)$.

Definition 2.5.31 (Direct product over a family of objects)**.** Let $\{c_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$. The *direct product* $\prod_{i\in I} c_i$ of $\{c_i\}$ is

1. an object in $\mathcal{C}$,
2. maps $\prod_{i\in I} c_i \to c_j$ for each $j \in I$ such that if there is an object c' and maps $c' \to c_j$ for each $j \in I$ then $\exists!$ map $c' \to \prod_{i\in I} c_i$ such that

$$\begin{tikzcd} c' \arrow[d, dashed] \arrow[dr, "if"] & \\ \prod_{i\in I} c_i \arrow[r, "given"'] & c_j \end{tikzcd}$$

   commutes for all $j \in I$.

Example 2.5.32. Let $\{V_i\}_{i\in\mathbb{N}} \subseteq \mathrm{Ob}(\mathbf{Vect}(X))$. Suppose $V_j = \mathbb{R}$ and consider the following diagram.

$$\begin{tikzcd} W \arrow[d, dashed, "\lambda"] \arrow[dr, "\psi_j"] & \\ \prod_{\mathbb{N}} V_i \arrow[r, "\theta_j"'] & V_j \end{tikzcd}$$

Note that θ_j must be surjective. One way to see it, is to pick $W = V_j$ and ψ_j to be the identity map. This implies that for any arbitrary set $\{v_j \in V_j : j \in I\}$, there exist a single element $w \in prod V_i$ such that $\theta_j(w) = v_j$, for all $j \in J$.

Guess: $\prod V_i = \{(v_1, v_2, v_3, \cdots) : v_i \in V_i\} = W$. We consider the following diagram

$$\begin{tikzcd} W \arrow[d, dashed, "\lambda"] \arrow[dr, "\psi_j"] & \\ \prod_{\mathbb{N}} V_i \arrow[r, "\theta_j"'] & V_j \end{tikzcd}$$

where the map ψ_j maps $(v_1, v_2, v_3, \cdots)$ to v_j. Now define the map $\gamma : \prod V_i \to W$ by $\gamma(x) = (\theta_1(x), \theta_2(x), \cdots)$. This implies that $\gamma = \lambda^{-1}$ and therefore $prod V_i = \{(v_1, v_2, v_3, \cdots)$.

Corollary 2.5.33. *If* $|\, I \,| < \infty$, *then* $\sqcup_I V_i = \sqcap_I V_i$ *for* $\{V_i\} \in \mathrm{Ob}(\mathbf{Vect}(X))$.

Note that this is not true for $\mathrm{Ob}(\mathbf{Open}(X))$.

Here I is used to index objects in $\mathcal{C}$. The pair $(\{c_i\}, I)$ can be thought of as a discrete category

$$I = \begin{tikzcd} \bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \bullet \arrow[loop above] & \cdots \end{tikzcd} .$$

with a functor $F : I \to \mathcal{C}$, mapping i to C_i. **Question**: What happens when I has more arrows?

Example 2.5.34. Take I to be the following category

$$I = \begin{tikzcd} \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \arrow[r] & \bullet \arrow[loop above] \end{tikzcd}$$

and let $F : I \to \mathbf{Open}(X)$. Note that $\bigcap_I F(i) = F(1)$ because $F(1) \subseteq F(2) \subseteq F(3)$.

2.5.2 Limits

Definition 2.5.35 (Limit of a functor)**.** Let I be a small category. The *limit* of a functor $F : I \to \mathcal{C}$ is

1. an object $\varprojlim F \in \mathcal{C}$,
2. maps $\varprojlim F \to F(i)$ for each $i \in \mathrm{Ob}(I)$ such that:
   - If $i \to j \in \mathrm{Mor}(I)$, then

$$\begin{tikzcd} F(i) \arrow[r] & F(j) \\ \varprojlim F \arrow[u] \arrow[ur] & \end{tikzcd}$$

   commutes.
   - If $c \to F(i) \in \mathrm{Mor}(\mathcal{C})$ for each $i \in I$ then there exists a unique map $c \to \varprojlim F$ such that:

$$\begin{tikzcd} F(i) \arrow[rr] & & F(j) \\ & \varprojlim F \arrow[ul, "given"'] \arrow[ur, "given"] & \\ & c \arrow[uul, "if"] \arrow[uur, "if"'] \arrow[u, dashed, "\exists!"'] & \end{tikzcd}$$

commutes whenever $i \to j \in \mathrm{Mor}(I)$.

Example 2.5.36.
Let I be the following category.

$$\begin{tikzcd} a & b \arrow[l] \arrow[d] \\ & c \end{tikzcd}$$

Let $F : I \to \mathbf{Vect}$. Then $\varprojlim F = F(b)$.

Example 2.5.37. Let I be the following category.

$$\begin{tikzcd} a \arrow[r, "f"] & b \\ & c \arrow[u, "g"'] \end{tikzcd}$$

Let $F : I \to \mathbf{Vect}$. Then $\varprojlim F = \{(v, w) \in F(a) \oplus F(c) : F(f)(v) = F(g)(w) \in F(b)\}$. Visually, $\varprojlim F$ fills out the commutative diagram

$$\begin{tikzcd} F(a) \arrow[r] & F(b) \\ \varprojlim F \arrow[u] \arrow[r] & F(c) \arrow[u] \end{tikzcd}$$

Example 2.5.38. Let $U, V \in \mathbf{Open}(X)^{op}$ and let $F : \mathbf{Open}(X)^{op} \to \mathbf{Set}$ be a functor. Let I be the following category.

$$\begin{tikzcd} U & & V \\ & U \cap V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Then $\varprojlim F = F(U \cup V)$ and satisfies the following commutative diagram.

$$
\begin{tikzcd}
 & \varprojlim F = F(U \cup V) & \\
F(U) \arrow[ur] & & F(V) \arrow[ul] \\
 & F(U \cap V) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Example 2.5.39. Take I to be the following category

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

and let $F : I \to \mathbf{Vect}(X)$. Then that $\varprojlim F = F(1)$.

Let I be the category where the objects are the non-positive integers with a morphisms from a to b if a is smaller than b. Let $F : I \to \mathbf{Vect}$ be a functor. The limit of F is $\varprojlim F = \{(\cdots, v_{-2}, v_{-1}, v_0) \in \prod F(i) : v_i = f_{i+1,i} \circ \cdots \circ f_{j,j-1}(v_j)\}$. Suppose

$$F_i = \begin{cases} \mathbb{R} & \text{if } i \text{ is odd} \\ \mathbb{R} \oplus \mathbb{R} & \text{if } i \text{ is even} \end{cases}$$

where $x \in \mathbb{R}$ is mapped to $(x, 0) \in \mathbb{R} \oplus \mathbb{R}$ and $(x, y) \in \mathbb{R} \oplus \mathbb{R}$ is mapped to $(x, 0) \in \mathbb{R}$. The limit of F is $\mathbb{R}$.

Example 2.5.40. Let I be the following category:

$$I = \begin{cases} \mathrm{Ob}(I) = \bullet \\ \mathrm{Mor}(I) = \mathbb{Z} & \text{with } a \circ b = a + b \end{cases}$$

Let $F : I \to \mathbf{Vect}$ be a functor defined on the object by $F(\bullet) = V$ and on the morphisms by $F(0) = \mathrm{id} : V \to V$, $F(1) = M : V \to V$. What is $\varprojlim F$?

Observation Let's consider the usual commutative diagram of a limit in our setting.

$$
\begin{tikzcd}
V \arrow[rr, "M^n"] & & V \\
 & \varprojlim F \arrow[ul, "\phi"] \arrow[ur, "\phi"'] & \\
 & z \arrow[uul, "\psi"] \arrow[uur, "\psi"'] \arrow[u, dashed] &
\end{tikzcd}
$$

We have $\psi(z) = M^n \psi(z)$, which implies that $\psi(z)$ belongs to the $+1$ eigenspace of M.

Guess: $\varprojlim F = +1$ eigenspace of M, which we will denote by E. Consider the following diagram.

$$
\begin{tikzcd}
V \arrow[rr, "M^n"] & & V \\
 & \varprojlim F \arrow[ul, "\phi"] \arrow[ur, "\phi"'] & \\
 & E \arrow[uul, "\imath"] \arrow[uur, "\imath"'] \arrow[u, dashed, "\lambda"] &
\end{tikzcd}
$$

By a similar, note that $\mathrm{im}(\phi) \subseteq E$, which implies that $\phi = \lambda^{-1}$. Therefore,

$$
\begin{aligned}
\varprojlim F &= \{v \in V : Mv = v\} \\
&= +1 \text{ eigenspace of } M^n \\
&= M \text{ invariants of } V.
\end{aligned}
$$

The limit of the functor F is therefore the set of $\mathbb{Z}$-fixed points. On the other hand, the colimit of the functor F is the set of $\mathbb{Z}$-orbits, but we will omit the proof here.

Example 2.5.41. We will conclude this lecture with a slight detour into p-adic numbers. Let I be the category where $\mathrm{Ob}I = \mathbb{Z}_{>0}$ and $a \to b$ if $b \leq a$. Let p be a prime number, and define the functor

$$\begin{aligned} &F : I \to \mathbf{Ab} \\ &\quad F(i) = \mathbb{Z}/p^i\mathbb{Z} \\ &\quad F(n+1 \to n)(x) = x \mod p^n \end{aligned}$$

We illustrate the image of F as:

$$\cdots \to \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{\text{mod } p^2} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{Z}/p\mathbb{Z}$$

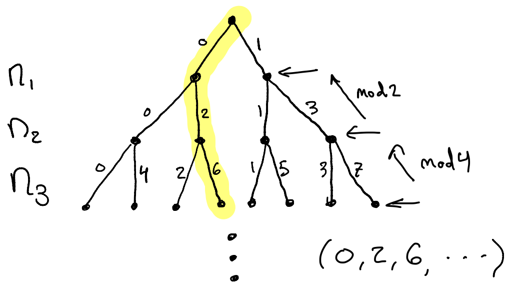
Question: $\varprojlim F = ?$
Solution:

$$\varprojlim F = \{(\cdots, n_3, n_2, n_1) : n_i \in \mathbb{Z}/p^i\mathbb{Z} \text{ such that if } j < k \text{ then } n_j \equiv n_k \mod p^j\}$$

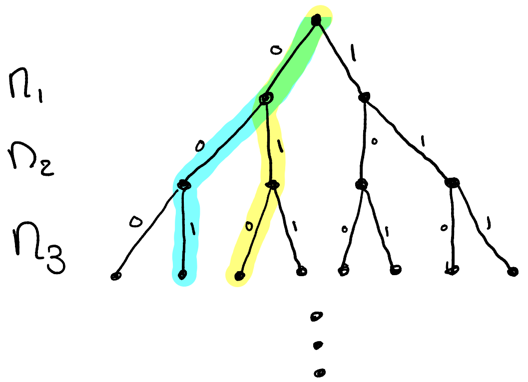
For concreteness, let $p = 2$. Then

$$\begin{aligned} (\cdots, n_3, n_2, n_1) &\in \varprojlim F \\ \text{if } n_2 \equiv n_1 &\mod 2 \\ n_3 \equiv n_2 &\mod 4 \\ \text{etc} \end{aligned}$$

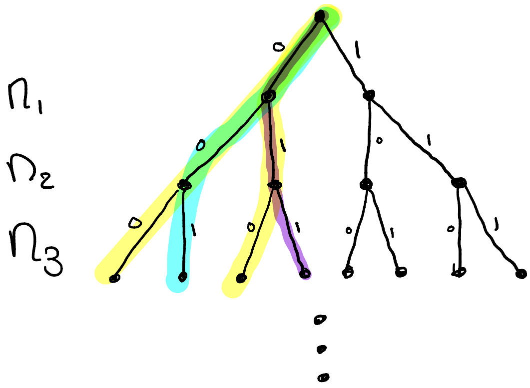
So, $(\cdots, 7, 3, 1) \in \varprojlim F$ and $(\cdots, 3, 2, 1, 0) \notin \varprojlim F$. We can visualize elements of $\varprojlim F$ as a tree:



where the highlighted path corresponds to the element $(\cdots, 6, 2, 0)$ in the notation above. We can also label branches of the tree using infinite strings of 0 and 1:



From this illustration, we can see that $\varprojlim F$ has a natural metric structure which is related to the node in which two branches connect. Moreover, one can show that $\varprojlim F$ has the structure of a field, which is called the field of 2-adic numbers (or for general primes, p-adic numbers, $\mathbb{Q}_p$). Unlike fields like $\mathbb{Q}$ and $\mathbb{R}$, the p-adic numbers admit several strange properties. For example, $\mathbb{Q}_p$ is compact (think of how to prove this using the tree visualization above), and every point in an interval is the center of the interval. To see the second claim, imagine highlighting all of the paths in the tree which are a fixed distance away from $(\cdots, 0, 0, 0)$ (illustrated below). Do this again starting with an element which you highlighted (for example $(\cdots, 0, 1, 0)$).



3 Homological Algebra

Lecture 6: Abelian category, exact sequence, derived functor

This lecture is about homological algebra, as a motivation for sheaf cohomology. One can think of homological algebra as the category theory generalization of algebraic topology. Like often in category theory, we will take a tool that is natural and powerful in one category—in our case (singular) (co-)homology in the category **Open**(X) for any topological space X—and we will express it in terms of category theory only, to be able to generalize it to other categories. The result is homological algebra. However, the drawback is that the thus defined objects are not defined constructively but by existence of an object of certain properties. So it is easy to define, but not easy to compute. For this reason we will later learn about sheaf cohomology, which is not as general, but by being more specific it will be easier to compute.

References. [KS13, Chapter 1 Section 1–3] and [Rot08, Sections 5.5, 3.1, 3.2, 6.2]

Singular (co-)homology in Open(X)

We first look at singular homology. There we start with the complex

$$S_{n+1} \to S_n \to \cdots \to S_0$$

consisting of the free abelian groups S_n generated by all singular simplices, that is all continuous maps from the standard n-simplex to X. The free abelian groups are connected by boundary maps. To specify the coefficient group G, we apply the functor $-\otimes_{\mathbb{Z}} G$ (which replaces the free abelian groups by a direct sum of copies of G, namely one copy of G per singular simplex):

$$S_{n+1} \otimes_{\mathbb{Z}} G \to S_n \otimes_{\mathbb{Z}} G \to \cdots \to S_0 \otimes_{\mathbb{Z}} G$$

And we get singular homology with coefficients in G by taking kernels mod images of the this new chain. To summarize, we start with a chain of maps, then we apply a functor, and we get homology by kernels mod images. (Note that the functor step could have easily been hidden by building the coefficient group already into the first step.)

In order to get singular cohomology, we follow the same recipe, but just apply a different functor, namely $\mathrm{Hom}(-,G)$. Note, that $\mathrm{Hom}(-,G)$ is contravariant and therefore flips the direction of the complex:

$$\mathrm{Hom}(S_{n+1},G) \leftarrow \mathrm{Hom}(S_n,G) \leftarrow \cdots \leftarrow \mathrm{Hom}(S_0,G)$$

Kernels and Images

In order to define homology as kernels mod images, we need to understand what kernels and images are in terms of category theory.

In the above setting we compute kernels and images in the category **Ab**. In this category, we can define the kernel of a group homomorphism $f : H \to G$ as $\ker f = \{h \in H \mid f(h) = 0\}$, where 0 is the identity element. To generalize this, we need a category theoretic 'zero'. This motivates the following definition, which can be seen as an enhancement of the definition of a locally small category, by asking for the morphisms between two objects to form not only a set, but even a group.

Definition 3.6.1 (Additive Category)**.** An *additive* category is a category $\mathcal{C}$ such that

1. $\mathrm{Hom}_{\mathcal{C}}(a,b)$ is an abelian group for each $a,b \in \mathrm{Ob}(\mathcal{C})$ and composition is bilinear:

$$(f+f')\circ(g+g') = f\circ g + f'\circ g + f\circ g' + f'\circ g'.$$

2. There is a 'zero object' $0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(0,0) = 0$ is the trivial group.

3. Finite products and coproducts[7] (i.e. the indexing set is finite) of objects in $\mathcal{C}$ exist.

Exercise 3.6.2. *If $\mathcal{C}$ is an additive category, then $a \bigsqcup b \simeq a \prod b$.*

Proof. Plug in the definitions. □

Hence, **Open**(X) is not additive, because coproducts (unions) and products (intersections) are different. This can alternatively be seen the following way.

Examples of (not) additive categories.

1) Let $\mathcal{P}$ be a (non-trivial) poset. Then the category $\mathcal{P}$ (in particular **Open**(X)) is <u>not</u> additive, because $\mathrm{Hom}_{\mathcal{P}}(a,b) = \varnothing$ for $a \nleq b$ and the empty set is not an abelian group because it misses an identity element.

2) The category **Set** is <u>not</u> additive, because $\mathrm{Hom}_{\mathbf{Set}}(X,\varnothing) = \varnothing$ for any non-empty set X.

3) **Ab** is an additive category, because group homorphisms can be added element-wise yielding the structure of an abelian group. Compositions are bilinear where linearity in the right component comes from the group homomorphism property and linearity in the left component comes from the definition through element-wise addition. The zero-object is the trivial group. Finite products and finite coproducts are both direct sums.

4) **Vect** is an additive category as well, for the same reasons as **Ab**.

5) Let $\mathcal{C}$ be any category (not necessarily additive). The category $\mathrm{Fun}(\mathcal{C},\mathbf{Ab})$ is additive, because $\mathrm{Hom}_{\mathrm{Fun}}(F,G)$, the set of natural transformations between two functors F,G, becomes a group through element-wise addition: The sum of $\tau,\sigma \in \mathrm{Hom}_{\mathrm{Fun}}(F,G)$ is defined as $(\tau+\sigma)(c) = \tau(c)+\sigma(c) \in \mathrm{Hom}_{\mathbf{Ab}}(F(c),G(c))$. The rest of the properties can be checked, making use of the fact that **Ab** is an additive category.

Next, we define functors that preserve the extra structure of additive categories.

Definition 3.6.3 (Additive Functor)**.** A functor $F:\mathcal{B}\to\mathcal{C}$ between additive categories is *additive* iff

$$\mathrm{Hom}_{\mathcal{B}}(a,b) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(F(a),F(b))$$

is a group homomorphism for each pair $a,b \in \mathrm{Ob}(\mathcal{B})$.

Assume $\mathcal{C}$ is an additive category. With this additional structure, we can define the kernel of a morphism.

Definition 3.6.4 (Kernel)**.** If $a \xrightarrow{f} b \in \mathrm{Mor}(\mathcal{C})$, then the *kernel* of f, $\ker f$, is (if it exists)

1. an object $\ker f \in \mathrm{Ob}(\mathcal{C})$

2. a morphism $\ker f \to a \in \mathrm{Mor}(\mathcal{C})$

such that

[7]Exercise 3.6.2 suggests that it is enough to require only one of the two to exist. This is indeed the case in this setting: Requiring that finite products exist will imply that finite coproducts exist and vice versa.

$$\begin{tikzcd}
a \arrow[r, "f"] & b \\
\ker f \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

commutes, and if there exists another object c together with a morphism $c \to a$ that makes

$$\begin{tikzcd}
a \arrow[r, "f"] & b \\
c \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

commute, then this morphism factors through $\ker f$, i.e. $\exists!\ c \dashrightarrow \ker f$ such that

$$\begin{tikzcd}
a \arrow[r, "f"] & b \\
\ker f \arrow[u] \arrow[ur, "0"'] & \\
c \arrow[u, dashed, "\exists!"'] \arrow[uu, bend left] \arrow[uur, bend right, "0"'] &
\end{tikzcd}$$

commutes.

By dualizing, we can define the cokernel:

Definition 3.6.5 (Cokernel)**.** If $a \xrightarrow{f} b \in \mathrm{Mor}(\mathcal{C})$, then the *cokernel* of f, $\operatorname{coker} f$, is (if it exists)

1. an object $\operatorname{coker} f \in \mathrm{Ob}(\mathcal{C})$
2. a morphism $\operatorname{coker} f \leftarrow b \in \mathrm{Mor}(\mathcal{C})$

such that

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "0"'] & b \arrow[d] \\
& \operatorname{coker} f
\end{tikzcd}$$

commutes, and if there exists another object c together with a morphism $c \leftarrow b$ that makes

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "0"'] & b \arrow[d] \\
& c
\end{tikzcd}$$

commute, then this morphism factors through $\operatorname{coker} f$, i.e. $\exists!\ c \dashleftarrow \operatorname{coker} f$ such that

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "0"'] \arrow[ddr, bend right, "0"'] & b \arrow[d] \arrow[dd, bend left] \\
& \operatorname{coker} f \arrow[d, dashed, "\exists!"'] \\
& c
\end{tikzcd}$$

commutes.

Intuitively, one can think of the cokernel as the part of the target space that is not reached by the function. In the category **Ab** (or **Vect**), the cokernel of a group homomorphism $A \xrightarrow{f} B$ is $\operatorname{coker} f = B \big/ \operatorname{im} f$.

Using the definition of kernel and cokernel, we can define image and coimage. Note that in the category **Ab**, the coimage of a group homomorphism $A \xrightarrow{f} B$ is $\operatorname{coim} f = A \big/ \ker f$.

Definition 3.6.6 (Coimage)**.** Let $a \xrightarrow{f} b \in \operatorname{Mor}(\mathcal{C})$ be a morphism. If $\ker f$ exists, let α denote the morphism $\ker f \xrightarrow{\alpha} a$ from the definition of kernel. The *coimage* of f is $\operatorname{coim} f := \operatorname{coker} \alpha$ (if it exists).

Dualizing yields the definition of image:

Definition 3.6.7 (Image)**.** Let $a \xrightarrow{f} b \in \operatorname{Mor}(\mathcal{C})$ be a morphism. If $\operatorname{coker} f$ exists, let β denote the morphism $\operatorname{coker} f \xleftarrow{\beta} b$ from the definition of cokernel. The *image* of f is $\operatorname{im} f = \ker \beta$ (if it exists).

Assuming $\ker f, \operatorname{coker} f, \operatorname{im} f, \operatorname{coim} f$ exist, we get the following commuting diagram:

$$
\begin{tikzcd}
\ker f \arrow[r, "\alpha"] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & a \arrow[r, "f"] \arrow[d] \arrow[rr, bend left, "0"] & b \arrow[r, "\beta"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[ur, dashed] \arrow[r, dashed] & \operatorname{im} f \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}
$$

where the solid arrows come from the definitions of $\ker f, \operatorname{coker} f, \operatorname{im} f, \operatorname{coim} f$. The dashed diagonal arrow comes from the definition of $\operatorname{coim} f$ as $\operatorname{coker} \alpha$. To induce the horizontal dashed arrow we first need to prove that $\gamma := \beta \circ (\operatorname{coim} f \dashrightarrow b)$ is 0. By the commutativity of the diagram, we see that $\gamma \circ (a \to \operatorname{coim} f) = 0$, yielding the following commutative diagram:

$$
\begin{tikzcd}
\ker f \arrow[r, "\alpha"] \arrow[dr, "0"'] \arrow[ddr, bend right, "0"'] & a \arrow[d] \arrow[dd, bend left=60, "0"] \\
& \operatorname{coim} f \arrow[d, bend right, "\gamma"'] \arrow[d, dashed, "\exists!"'] \arrow[d, bend left, "0"] \\
& \operatorname{coker} f
\end{tikzcd}
$$

Using the definition of $\operatorname{coim} f$ as $\operatorname{coker} \alpha$, we see that there must be a unique arrow $\operatorname{coim} f \dashrightarrow \operatorname{coker} f$ which makes the diagram commute. Therefore $\gamma = \beta \circ (\operatorname{coim} f \dashrightarrow b) = 0$. Finally, we obtain the horizontal dashed arrow $\operatorname{coim} f \dashrightarrow \operatorname{im} f$ from the definition of $\operatorname{im} f$ as $\ker \beta$.

Note that in the the category **Ab** for a group homomorphism $A \xrightarrow{f} B$, this horizontal dashed arrow is the isomorphism between $\operatorname{coim} f = A/\ker f$ and $\operatorname{im} f$ from the first isomorphism theorem. This motivates the following definition.

Definition 3.6.8 (Abelian Category)**.** An additive category $\mathcal{C}$ is *abelian* iff

1. $\ker f$ and $\operatorname{coker} f$ exist[8] for each $f \in \operatorname{Mor}(\mathcal{C})$
2. the canonical morphism $\operatorname{coim} f \dashrightarrow \operatorname{im} f$ is an ismorphism for each $f \in \operatorname{Mor}(\mathcal{C})$.

Intuition: "Abelian categories behave like the category of abelian groups."

Examples of (not) abelian categories.

1) **Ab** is an abelian category because it is additive (see the 3rd example of (non-)additive categories), $\ker f$ and $\operatorname{coker} f = B/\operatorname{im} f$ exist, and the first isomorphism theorem yields an isomorphism between $\operatorname{coim} f = A/\ker f$ and $\operatorname{im} f$.

2) **Vect** is an abelian category as well, for the same reasons as **Ab**.

[8]and thus also $\operatorname{im} f$ and $\operatorname{coim} f$ exist as they are defined as kernels and cokernels

3) Let $\mathcal{C}$ be any category (not necessarily additive or abelian). The category $\text{Fun}(\mathcal{C}, \mathbf{Ab})$ is abelian[9], because it is additive (see the 5th example of (non-)additive categories) and the rest of the properties can be checked, making use of the fact that $\mathbf{Ab}$ is an abelian category. In particular, for any poset $\mathcal{P}$, $\text{Fun}(\mathcal{P}, \mathbf{Ab})$ is abelian. This motivates sheaf theory: $\mathbf{Open}(X)$ does not have the necessary structure to use algebra, but $\text{Fun}(\mathbf{Open}(X), \mathbf{Ab})$ does.

4) So far, all examples that were additive were also abelian. However, there exist also examples of additive categories that are not abelian, for example $\mathbf{Ban}_{\mathbb{C}}$, the category of complex Banach spaces with continuous linear maps. However, in order to see why it is not abelian, we would need functional analysis.

Exact Sequences

For the rest of this lecture, assume $\mathcal{C}$ is abelian. This extra structure allows to study complexes of objects (like when computing (co-)homology).

Definition 3.6.9 (Exact Pair of Composable Morphisms)**.** A pair of composable morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{C}$ is *exact* iff

1. $g \circ f = 0$
2. The induced morphism $\operatorname{im} f \dashrightarrow \ker g$ is an isomorphism.[10]

$$
\begin{tikzcd}
& & & & \operatorname{coker} f \arrow[d, dashed] \\
a \arrow[rr, "f"] \arrow[rrrr, bend left, "0"] & & b \arrow[rr, "g"] \arrow[rru] & & c \\
& \operatorname{im} f \arrow[ur] \arrow[rr, dashed] \arrow[rrruu, bend left, "0"] & & \ker g \arrow[ul] &
\end{tikzcd}
$$

where the vertical dashed arrow is induced by the definition of $\operatorname{coker} f$ and the fact that $g \circ f = 0$; and the horizontal dashed arrow is induced by the definition of $\ker g$ and the fact that the composition of g with the morphism of $\operatorname{im} f$ is zero.

Definition 3.6.10 (Exact Sequence)**.** A sequence

$$\dots \longrightarrow a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_{i+1}} a_{i+2} \longrightarrow \dots$$

in $\mathcal{C}$ is *exact* iff each pair of composable morphisms $a_i \xrightarrow{f_i} a_{i+1} \xrightarrow{f_{i+1}} a_{i+2}$ is exact.

Definition 3.6.11 (Left Exact Functor)**.** An additive functor $F : \mathcal{B} \to \mathcal{C}$ between abelian categories is *left exact* iff for each exact sequence[11]

$$0 \longrightarrow a \longrightarrow b \longrightarrow c$$

in $\mathcal{B}$, the sequence

$$0 \longrightarrow F(a) \longrightarrow F(b) \longrightarrow F(c)$$

is exact in $\mathcal{C}$.

Intuition: "Left exact functors preserve injectivity"

[9] And it is true for any abelian category $\mathcal{D}$, not just $\mathbf{Ab}$ itself, that $\text{Fun}(\mathcal{C}, \mathcal{D})$ is abelian.

[10] In $\mathbf{Vect}$ or $\mathbf{Ab}$ these two conditions simplify to one: $\operatorname{im} f = \ker g$.

[11] Sometimes, in the definition of left exactness a 0 is added at the end of the sequence, i.e. left exact iff $0 \to a \to b \to c \to 0$ exact $\implies 0 \to F(a) \to F(b) \to F(c)$ exact. This definition is however equivalent [?].

Definition 3.6.12 (Right Exact Functor)**.** An additive functor $F : \mathcal{B} \to \mathcal{C}$ between abelian categories is *right exact* iff for each exact sequence

$$a \longrightarrow b \longrightarrow c \longrightarrow 0$$

in $\mathcal{B}$, the sequence

$$F(a) \longrightarrow F(b) \longrightarrow F(c) \longrightarrow 0$$

is exact in $\mathcal{C}$.

Intuition: "Right exact functors preserve surjectivity"

Definition 3.6.13 (Exact Functor)**.** An additive functor $F : \mathcal{B} \to \mathcal{C}$ between abelian categories is *exact* iff it is right exact and left exact, i.e. iff[12] for each exact sequence

$$0 \longrightarrow a \longrightarrow b \longrightarrow c \longrightarrow 0$$

in $\mathcal{B}$, the sequence

$$0 \longrightarrow F(a) \longrightarrow F(b) \longrightarrow F(c) \longrightarrow 0$$

is exact in $\mathcal{C}$.

From an algebraic perspective, (co-)homology is a way to measure how badly a functor fails to be exact.

Example 3.6.14. Let $\mathcal{C}$ be an abelian category. For each $c \in \mathrm{Ob}(\mathcal{C})$, the functor

$$\mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathbf{Ab}$$

is left exact, but not necessarily right exact. For example for $\mathcal{C} = \mathbf{Ab}$, the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/_{2\mathbb{Z}} \longrightarrow 0$$

is exact, but if we apply $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/_{2\mathbb{Z}}, -)$ we get (up to isomorphism)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/_{2\mathbb{Z}} \longrightarrow 0$$

which cannot be exact, because the kernel of the last map is $\mathbb{Z}/_{2\mathbb{Z}}$ but the image of the map before this is 0, and hence not the same. This shows that $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/_{2\mathbb{Z}}, -)$ is not right exact. Using the following definition we can therefore say that $\mathbb{Z}/_{2\mathbb{Z}}$ is not projective.

Definition 3.6.15 (Projective and Injective Objects)**.** An object $c \in \mathrm{Ob}(\mathcal{C})$ is *projective* iff $hom_{\mathcal{C}}(c, -)$ is exact, and respectively *injective* iff $hom_{\mathcal{C}}(-, c)$ is exact.

Definition 3.6.16 (Projective and Injective Resolutions)**.** Let $\mathcal{C}$ be an abelian category and $c \in \mathrm{Ob}(\mathcal{C})$.

A *projective resolution of* c is (if it exists) an exact sequence

$$\dots \to P_2 \to P_1 \to P_0 \to c \to 0$$

where P_i is a projective object in $\mathcal{C}$ for each i.

An *injective resolution of* c is (if it exists) an exact sequence

$$0 \to c \to I_0 \to I_1 \to I_2 \to \dots$$

where I_i is an injective object in $\mathcal{C}$ for each i.

It is very difficult to come up with projective or injective resolutions, as it is already difficult to come up with projective/injective objects. Sheaf theory is useful because we will use topology to build these sequences instead of having to guess them.

[12]The implication from right to left is not obvious with the definitions we use. However, with the equivalent definition of left and right exactness—explained in the footnote in the definition of left exactness—it becomes obvious.

Left Derived Functors

Suppose $F : \mathcal{B} \to \mathcal{C}$ is a right exact additive functor between abelian categories. Suppose each object in $\mathcal{B}$ has a projective resolution.

Then to each object $b \in \mathrm{Ob}(\mathcal{B})$ we can choose a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to b \to 0$$

and apply F to get a sequence

$$\ldots \xrightarrow{d_3} F(P_2) \xrightarrow{d_2} F(P_1) \underbrace{\xrightarrow{d_1} F(P_0) \xrightarrow{\varepsilon} F(b) \to 0}_{\text{exact}}$$

The left part of the sequence may no longer be exact because the functor is only right exact.

Now consider the sequence where the object $F(b)$ is removed and only the projective objects are left

$$\ldots \xrightarrow{d_3} F(P_2) \xrightarrow{d_2} F(P_1) \xrightarrow{d_1} F(P_0) \xrightarrow{d_0} 0$$

Even though the exactness might not be preserved, the property of two consecutive maps composing to 0 is preserved by any additive functor. We therefore have the induced morphisms $\operatorname{im} d_{n+1} \dashrightarrow \ker d_n$ from Definition 3.6.9. These morphisms do not need to be isomorphims, and intuitively we can measure how far they are from being isomorphisms by their cokernels. We thus define, the *left derived functor* of F as

$$(L_n F)(b) \coloneqq \operatorname{coker}(\operatorname{im} d_{n+1} \dashrightarrow \ker d_n).$$

Note that in the category **Vect** or **Ab** we would have $\operatorname{im} d_{n+1} \subseteq \ker d_n$ and we would define $(L_n F)(b) \coloneqq {}^{\ker d_n}\!/_{\operatorname{im} d_{n+1}}$. However, in other categories the notion of quotient is not defined, so we define it in terms of cokernels.

The left derived functor is the analogue of homology.

Example 3.6.17. We prove $(L_0 F)(b) \simeq F(b)$. The proof holds in general, but is easier to read in the category $\mathcal{C} = \mathbf{Ab}$ or $\mathcal{C} = \mathbf{Vect}$: As F is right exact we have $\operatorname{im} d_1 = \ker \varepsilon$ and ε surjective. And hence,

$$(L_0 F)(b) = {}^{\ker d_0}\!/_{\operatorname{im} d_1} = {}^{F(P_0)}\!/_{\operatorname{im} d_1} = {}^{F(P_0)}\!/_{\ker \varepsilon} \simeq \operatorname{im} \varepsilon = F(b).$$

For general abelian categories $\mathcal{C}$ the proof follows the same principle but looks more complicated:

$$(L_0 F)(b) = \operatorname{coker}(\operatorname{im} d_1 \dashrightarrow \ker d_0) \simeq \operatorname{coker} d_1 \simeq \operatorname{coim} \varepsilon \simeq \operatorname{im} \varepsilon \simeq F(b)$$

where the first isomorphism can be proven using $\ker d_0 \simeq F(P_0)$ (as $d_0 = 0$), the second using $\operatorname{im} d_1 \simeq \ker \varepsilon$ (because F is right exact), the third isomorphism is given by the definition of $\mathcal{C}$ abelian and the last isomorphism is $\operatorname{im} \varepsilon \simeq \ker 0 \simeq F(b)$ (because F is right exact).

The notation of the left derived functor as $(L_n F)(b)$ suggests that the constructions does not depend on the choice of projective resolution. And this is true as the following theorem shows.

Theorem 3.6.18. *With all above assumptions*

1. $(L_n F)(b)$ *does not depend on the choice of projective resolution of* b.
2. $L_n F$ *defines an additive functor*[13] *from* $\mathcal{B}$ *to* $\mathcal{C}$.
3. *If*

$$0 \longrightarrow a \longrightarrow b \longrightarrow c \longrightarrow 0$$

is exact in $\mathcal{B}$, *then*

[13] We only defined $L_n F$ on objects, not on morphsisms. To define $L_n F$ on a morphism, one would need to lift the morphism to the projective resolutions (not necessarily uniquely) which would then induce a morphism between the corresponding cokernels (which is unique).

$$\begin{tikzcd}
\dots \arrow[r] & (L_nF)(a) \arrow[r] & (L_nF)(b) \arrow[r] & (L_nF)(c) \arrow[dll] \\
 & (L_{n-1}F)(a) \arrow[r] & (L_{n-1}F)(b) \arrow[r] & (L_{n-1}F)(c) \arrow[dll] \\
 & \dots & \dots & \dots \arrow[dll] \\
 & (L_1F)(a) \arrow[r] & (L_1F)(b) \arrow[r] & (L_1F)(c) \arrow[dll] \\
 & F(a) \arrow[r] & F(b) \arrow[r] & F(c) \arrow[r] & 0
\end{tikzcd}$$

is exact in $\mathcal{C}$.

We will now dualize everything to get right derived functors.

Right Derived Functors

Suppose $F : \mathcal{B} \to \mathcal{C}$ is a left exact additive functor between abelian categories. Suppose each object in $\mathcal{B}$ has an injective resolution.

Then to each object $b \in \mathrm{Ob}(\mathcal{B})$ we can choose an injective resolution

$$0 \to b \to I_0 \to I_1 \to I_2 \to \dots$$

and apply F to get a sequence

$$\underbrace{0 \longrightarrow F(b) \xrightarrow{\eta} F(I_0) \xrightarrow{\partial_1}}_{\text{exact}} F(I_1) \xrightarrow{\partial_2} F(I_2) \to \dots$$

The right part of the sequence may no longer be exact because the functor is only left exact. Removing $F(b)$ yields

$$0 \xrightarrow{\partial_0} F(I_0) \xrightarrow{\partial_1} F(I_1) \xrightarrow{\partial_2} F(I_2) \to \dots$$

Again, the property of two consecutive maps composing to 0 induces the morphisms $\operatorname{im} \partial_{n-1} \dashrightarrow \ker \partial_n$ from Definition 3.6.9. We measure how far they are from being isomorphisms by their cokernels (in **Vect** or **Ab**, by their quotients ${}^{\ker \partial_n}\!/\!_{\operatorname{im} \partial_{n-1}}$). We thus define, the *right derived functor* of F as

$$(R^nF)(b) \coloneqq \operatorname{coker}(\operatorname{im} \partial_{n-1} \dashrightarrow \ker \partial_n).$$

The right derived functor is the analogue of cohomology.

Example 3.6.19. $(R^0F)(b) \simeq F(b)$ can be proven using the fact that F is left exact.

Again, as the notation $(R^nF)(b)$ suggests, the constructions does not depend on the choice of injective resolution:

Theorem 3.6.20. *With all above assumptions*

1. *R^nF defines an additive functor*[14] *from $\mathcal{B}$ to $\mathcal{C}$.*
2. *R^nF does not depend on the choice of injective resolution (up to isomorphism).*
3. *If*

$$0 \longrightarrow a \longrightarrow b \longrightarrow c \longrightarrow 0$$

is exact in $\mathcal{B}$, then

[14]We only defined R^nF on objects, not on morphsisms. To define R^nF on a morphism, one would need to lift the morphism to the injective resolutions (not necessarily uniquely) which would then induce a morphism between the corresponding cokernels (which is unique).

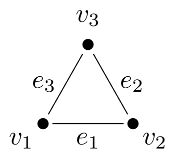
$$
\begin{tikzcd}
0 \arrow[r] & F(a) \arrow[r] & F(b) \arrow[r] & F(c) \arrow[dll] & \\
& (R^1F)(a) \arrow[r] & (R^1F)(b) \arrow[r] & (R^1F)(c) \arrow[dll] & \\
& \dots & \dots & \dots \arrow[dll] & \\
& (R^{n-1}F)(a) \arrow[r] & (R^{n-1}F)(b) \arrow[r] & (R^{n-1}F)(c) \arrow[dll] & \\
& (R^nF)(a) \arrow[r] & (R^nF)(b) \arrow[r] & (R^nF)(c) \arrow[r] & \dots
\end{tikzcd}
$$

is exact in $\mathcal{C}$.

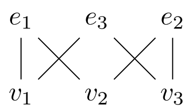
Lecture 7: Concrete computation of resolutions and derived functors

In this lecture we go through concrete examples of computing cohomology of simplicial complexes using the machinery introduced in the previous lecture.

Example 3.7.21. Throughout the lecture, we will work with a triangle as a simple object to illustrate all the constructions: let Σ be the simplicial complex as on the figure below (left). We can also see it as a partially ordered set (right).



(a) The simplicial complex Σ



(b) Σ as a poset

Choosing the category

To compute cohomology of Σ using the homological algebra, we first need to have an abelian category in which we have some representation of Σ as an object.[15] We can view Σ as its poset category, but that itself is neither abelian nor does it have Σ as an object in it. We take $\mathrm{Fun}(\Sigma, \mathbf{Ab})$, which is abelian, as our initial category — this is the first choice we make; it is a natural example, as such functor category will always be abelian no matter what Σ is.

Now we have two other choices to make. We need to choose

(a) a particular functor from Σ to $\mathbf{Ab}$ as the starting object representing Σ,

(b) a functor from $\mathrm{Fun}(\Sigma, \mathbf{Ab})$ to $\mathbf{Ab}$ which will yield a right derived functor[16] — we will need this to be an additive left exact functor.

For (a), the simplest possibility is to chose an abelian group G, and 'send everything to G':

$$
\begin{aligned}
\mathbb{1}_G : \Sigma &\longrightarrow \mathbf{Ab} \\
\sigma &\longmapsto G \\
\sigma \leq \tau &\longmapsto \mathrm{id}_G
\end{aligned}
$$

This group G will play the role of the coefficient group in the final cohomology.

For (b), we will look at two options that are always available. Both of these are also additive and left exact in general, which makes them good candidates to study:

[15] So that we can compute the injective resolution of that object (or weaker sufficient alternative of that), to which we then apply a left exact (additive) functor, and obtain the right derived functor, which finally yields (a version of) the cohomology.

[16] In the full generality, this could be a functor to any abelian category.

- The covariant Hom-functor for a fixed functor H. This is really a family of examples, since we have a choice of the functor H.

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(\Sigma,\mathbf{Ab})}(H,-) : \mathrm{Fun}(\Sigma,\mathbf{Ab}) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \mathrm{Hom}_{\mathrm{Fun}(\Sigma,\mathbf{Ab})}(H,F) \\ \eta : F \Rightarrow G &\longmapsto \eta_*\end{aligned}$$

  An object F of $\mathrm{Fun}(\Sigma,\mathbf{Ab})$, which is a functor, is sent to the set of natural transformations from H to F. This Hom-set has an additional structure of an abelian group inherited from $\mathbf{Ab}$: for $\nu,\mu : H \Rightarrow F$, the addition $\nu+\mu$ is defined component-wise as $(\nu+\mu)_x \coloneqq \nu_x + \mu_x$ for each object $x \in \mathrm{Ob}(\Sigma)$, where $\nu_x + \mu_x$ is defined by the abelian group structure of $\mathrm{Hom}_{\mathbf{Ab}}(H(x),F(x))$.

  A morphism of $\mathrm{Fun}(\Sigma,\mathbf{Ab})$, a natural transformation $\eta : F \Rightarrow G$, is sent to a mapping η_* acting via post-composing. That is, given a natural transformation $\nu : H \Rightarrow F$, we obtain a natural transformation $\eta_*(\nu) \coloneqq \eta \circ \nu : H \Rightarrow G$ (see vertical composition in 6.1.1).

- The limit functor.

$$\begin{aligned}\varprojlim : \mathrm{Fun}(\Sigma,\mathbf{Ab}) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \varprojlim F \\ \eta : F \Rightarrow G &\longmapsto \varprojlim F \xrightarrow{e} \varprojlim G\end{aligned}$$

  Since Σ is in our case a small category, we can see functors going from it as diagrams, and we can, therefore, consider limits. The category $\mathbf{Ab}$ has all limits, and limits are unique up to isomorphism, so $\varprojlim F$ will always be well defined.

  The morphisms are uniquely determined by the universal property of limits. The natural transformation $\eta : F \Rightarrow G$ yields a cone form $\varprojlim F$ to the diagram G, which, by the universal property of $\varprojlim G$, factors through $\varprojlim G$ via the unique morphism e.

Sanity check for the construction with the functors introduced above. We do not need the full resolution to calculate degree zero derived functors — as noted in the previous lecture, $R^0F(b) \cong F(b)$. Let us check what we get if we choose the functor $\mathbb{1}_{\mathbb{Z}} : \Sigma \to \mathbf{Ab}$ for (a), and the limit functor $\varprojlim : \mathrm{Fun}(\Sigma,\mathbf{Ab}) \longrightarrow \mathbf{Ab}$ for (b) for two different simplicial complexes Σ.

- First let $\Sigma = \bullet\ \ \bullet$ be a simplicial complex with just two vertices. Then Σ as poset-category has only two objects with no morphisms between them. The limit $\varprojlim \mathbb{1}_{\mathbb{Z}}$ is then a product of two copies of $\mathbb{Z}$:

$$R^0\varprojlim \mathbb{1}_{\mathbb{Z}} \cong \varprojlim \mathbb{1}_{\mathbb{Z}} = \Pi_\Sigma \mathbb{Z} = \mathbb{Z}\times\mathbb{Z} = \mathbb{Z}\oplus\mathbb{Z}$$

- Now let $\Sigma = \bullet\!\!-\!\!\bullet$ be a simplicial complex with two connected vertices. The corresponding poset category is $v_1 \to e \leftarrow v_2$, so the limit $\varprojlim \mathbb{1}_{\mathbb{Z}}$ is a pullback:

$$\begin{aligned}R^0\varprojlim \mathbb{1}_{\mathbb{Z}} \cong \varprojlim \mathbb{1}_{\mathbb{Z}} &= \{(n,m) \in \mathbb{1}_{\mathbb{Z}}(v_1)\oplus\mathbb{1}_{\mathbb{Z}}(v_2) \mid \mathbb{1}_{\mathbb{Z}}(v_1\to e)(n) = \mathbb{1}_{\mathbb{Z}}(v_2\to e)(m)\} \\ &= \{(n,m)\in\mathbb{Z}\oplus\mathbb{Z} \mid n = m\} \cong \mathbb{Z}\end{aligned}$$

We see that we got a free abelian group of the rank the number of connected components, which is what we would expect the zero cohomology with coefficients in $\mathbb{Z}$ to be.

Constructing the resolution

It is in general difficult to find the injective resolution needed for the definition of the right derived functors. We can see this as analogous to the difficulty of finding some simplicial structure on a topological space. It turns out that under some additional assumptions, we do not need to find injective resolution; a certain 'weaker resolution' suffices.

Back to the triangle Σ. Let us fix some abelian group G as the coefficient group, and let us fix the functor $\mathbb{1}_G : \Sigma \to \mathbf{Ab}$ as our representation of Σ in the category $\mathrm{Fun}(\Sigma, \mathbf{Ab})$.

Now we would like to construct an injective resolution of $\mathbb{1}_G$, that is, an exact sequence $0 \to \mathbb{1}_G \to I_0 \to I_1 \to \dots$, where each I_j is injective. However, the injectivity is difficult to compute or test algorithmically. Luckily, it turns out that in our more specific setting, it is not really needed. Together with the exactness, it is enough to assume surjectivity of certain maps in the construction. We need I_j's to be so called *flabby* (or *flasque*) sheaves. In particular, $I_j(f)$ needs to be surjective for every morphism f in Σ for all the functors $I_j : \Sigma \to \mathbf{Ab}$ in the resolution, but there are more conditions — we will not go into details at this point. We just state here that the following construction, while not necessarily injective, is sufficient for computing the right derived functor. The construction would be analogous for any simplicial complex Σ, but for simplicity, we will focus on the triangle.

Constructing I_0. We need to define a functor I_0, and a natural transformation $\mathbb{1}_G \to I_0$. We define I_0 as follows:

$$\begin{aligned} I_0 : \Sigma &\longrightarrow \mathbf{Ab} \\ \sigma &\longmapsto \bigoplus_{\sigma \le \tau} G \\ \gamma \le \sigma &\longmapsto \bigoplus_{\gamma \le \tau} G \xrightarrow{\mathrm{Proj}} \bigoplus_{\sigma \le \tau} G \end{aligned}$$

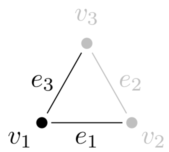
The direct sum $\bigoplus_{\sigma \le \tau} G$ goes over all cofaces of σ (in general of arbitrary dimensions). For example $\bigoplus_{v_1 \le \tau} G = \bigoplus_{\{v_1, e_1, e_3\}} G = G \oplus G \oplus G$. To define the morphisms, we see the copies of G as labeled by the simplices, and we just project to the corresponding coordinates. For example for $v_1 \le e_1$ we get

$$\begin{aligned} I_0(v_1 \le e_1) : \bigoplus_{\{v_1, e_1, e_3\}} G &\longrightarrow \bigoplus_{\{e_1\}} G \\ (g_1, g_2, g_3) &\longmapsto g_2 \end{aligned}$$

Note that the projections are all surjective, because if $\gamma \le \sigma$, then all the cofaces of σ are also cofaces of γ.

To define the natural transformation $\varepsilon : \mathbb{1}_G \to I_0$, we need to fill in the ε_σ's in the following diagram(s):

$$\begin{tikzcd} G = \mathbb{1}_G(e_1) \arrow[r, "\varepsilon_{e_1}"] & I_0(e_1) = G \\ G = \mathbb{1}_G(v_1) \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[r, "\varepsilon_{v_1}"] & I_0(v_1) = \underset{v_1}{G} \oplus \underset{e_1}{G} \oplus \underset{e_3}{G} \arrow[u, "\mathrm{Proj}_2"'] \arrow[d, "\mathrm{Proj}_3"] \\ G = \mathbb{1}_G(e_3) \arrow[r, "\varepsilon_{e_3}"] & I_0(e_3) = G \end{tikzcd}$$



We need it to commute, and we need $0 \longrightarrow \mathbb{1}_G \xrightarrow{\varepsilon} I_0$to be exact, which means that for each $\sigma \in \mathrm{Ob}(\Sigma)$, the sequence $0 \longrightarrow \mathbb{1}_G(\sigma) \xrightarrow{\varepsilon_\sigma} I_0(\sigma)$needs to be exact. That is, all ε_σ's must be injective. We define $\varepsilon_{e_j} := \mathrm{id}_G$ for all the edges. For ε_{v_1}, we know from the commutativity of the diagram that g needs to be sent to $(?, g, g)$. The "?" can be chosen arbitrarily — we choose it as 0, which will be useful later. That is, we define $\varepsilon_{v_1} : g \mapsto (0, g, g)$, and similarly for the other two vertices.

Constructing I_1. Next, we define functor I_1, and natural transformation $\partial_0 : I_0 \to I_1$. The functor I_1 will be very similar to I_0, but instead of direct sums over all cofaces, we take direct sums over all strict cofaces:

$$\begin{aligned} I_1 : \Sigma &\longrightarrow \mathbf{Ab} \\ \sigma &\longmapsto \bigoplus_{\sigma < \tau} G \\ \gamma \le \sigma &\longmapsto \bigoplus_{\gamma < \tau} G \xrightarrow{\text{Proj}} \bigoplus_{\sigma < \tau} G \end{aligned}$$

In our situation, this means that vertices will be sent to $G \oplus G$, and edges will be sent to 0. The morphisms $(\partial_0)_\sigma$ need to be defined so that we have exactness in I_0, i.e., so that their kernels are equal to the images of ε_σ.

$$\begin{tikzcd} G = \mathbb{1}_G(e_1) \arrow[r, "\varepsilon_{e_1}"] & I_0(e_1) = G \arrow[r, "(\partial_0)_{e_1}"] & I_1(e_1) = 0 \\ G = \mathbb{1}_G(v_1) \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[r, "\varepsilon_{v_1}"] & I_0(v_1) = \underset{v_1}{G} \oplus \underset{e_1}{G} \oplus \underset{e_3}{G} \arrow[u, "\mathrm{Proj}_2"'] \arrow[d, "\mathrm{Proj}_3"] \arrow[r, "(\partial_0)_{v_1}"] & I_1(v_1) = \underset{e_1}{G} \oplus \underset{e_3}{G} \arrow[u, "0"'] \arrow[d, "0"] \\ G = \mathbb{1}_G(e_3) \arrow[r, "\varepsilon_{e_3}"] & I_0(e_3) = G \arrow[r, "(\partial_0)_{e_3}"] & I_1(e_3) = 0 \end{tikzcd}$$

The morphisms of ∂_0 for edges are clearly all 0, since the codomain is 0 – this is also the only choice to obtain exactness in I_0 for the edges.

We need to define $(\partial_0)_{v_1}$ so that $\operatorname{im} \varepsilon_{v_1} = \ker (\partial_0)_{v_1}$, i.e., (g_1, g_2, g_3) is sent to 0 iff it is of the form $(0, g, g)$. One way to do this is to define $(\partial_0)_{v_1} : (g_1, g_2, g_3) \mapsto (g_1, g_3 - g_2)$. We proceed analogously for v_2, v_3.

Constructing I_2 and further. In our case, we see that ∂_0 is actually surjective. This means that we can define $I_k = 0$ for all $k \ge 2$, and the sequence

$$0 \longrightarrow \mathbb{1}_G \xrightarrow{\varepsilon} I_0 \xrightarrow{\partial_0} I_1 \longrightarrow 0$$

is exact. This is because we have no faces of dimension higher than one.

In general, we would proceed analogously, defining $I_2(\sigma)$ as the direct sum of as many copies of G as there are cofaces of σ with dimension higher by at least two. The maximal k for which I_k is non-zero is the dimension of the simplicial complex. The construction would also work for CW-complexes or similar nice objects.

Functors from $\operatorname{Fun}(\Sigma, \mathbf{Ab})$ to $\mathbf{Ab}$

With the resolution at hand, we can apply different additive left exact functors, and obtain possibly different notions of cohomology. Here we go through two examples that ultimately lead to the same cohomology. We look at representable functor $\operatorname{Hom}_{\operatorname{Fun}}(\mathbb{1}_\mathbb{Z}, -)$, and the limit functor $\varprojlim$, both of which we already briefly introduced before.

The representable functor $\operatorname{Hom}_{\operatorname{Fun}}(\mathbb{1}_\mathbb{Z}, -)$. The first functor we look at is

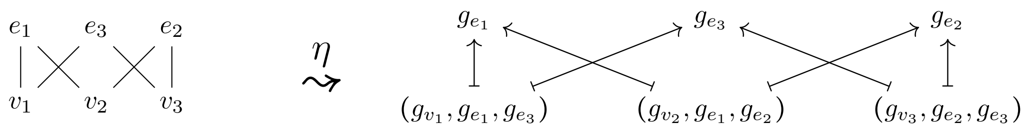
$$\begin{aligned} \operatorname{Hom}_{\operatorname{Fun}(\Sigma,\mathbf{Ab})}(\mathbb{1}_\mathbb{Z}, -) : \operatorname{Fun}(\Sigma, \mathbf{Ab}) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \operatorname{Hom}_{\operatorname{Fun}(\Sigma,\mathbf{Ab})}(\mathbb{1}_\mathbb{Z}, F) \\ \eta : F \Rightarrow G &\longmapsto \eta_* \end{aligned}$$

Choosing a representable functor is nice, since they are left exact in general. Choosing $\mathbb{1}_\mathbb{Z}$ for the domain is nice, since $\mathbb{Z}$ is the free Abelian group with one generator. This means that for any abelian group A, we can see any map $\mathbb{Z} \to A$ as choosing one element of A – the one where 1 is sent – and every element of A is represented by a unique such map. This yields a natural bijection $\operatorname{Hom}_{\mathbf{Ab}}(\mathbb{Z}, A) \cong A$. We can then view an element of $\operatorname{Hom}_{\operatorname{Fun}}(\mathbb{1}_\mathbb{Z}, F)$ as picking an element in each of the groups in the image of F. We apply this functor to the sequence $0 \to I_0 \xrightarrow{\partial_0} I_1 \to 0$ (recall that we remove $\mathbb{1}_G$ from the beginning of the resolution before applying the functor F).

First, we look at what $\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_0)$ is – what are all the natural transformations η from $\mathbb{1}_{\mathbb{Z}}$ to I_0? That is, what are all the possibilities to choose one element from each $I_0(\sigma)$ so that the choices are consistent with all the maps in I_0? Let us choose an element (a_1, a_2, a_3) from the group $I_0(v_1)$ and (b_1, b_2, b_3) from $I_0(v_2)$, and look at the following diagrams:

$$
\begin{tikzcd}
1 \arrow[r, mapsto] & (a_1, a_2, a_3) \\
\mathbb{Z} = \mathbb{1}_{\mathbb{Z}}(v_1) \arrow[r, "\eta_{v_1}"] \arrow[d, "\mathrm{id}"'] & I_0(v_1) = \underset{v_1}{G} \oplus \underset{e_1}{G} \oplus \underset{e_3}{G} \arrow[d, "\mathrm{Proj}_2"] \\
\mathbb{Z} = \mathbb{1}_{\mathbb{Z}}(e_1) \arrow[r, "\eta_{e_1}"] & I_0(e_1) = G \\
1 \arrow[r, mapsto] & a_2
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, mapsto] & (b_1, b_2, b_3) \\
\mathbb{Z} = \mathbb{1}_{\mathbb{Z}}(v_2) \arrow[r, "\eta_{v_2}"] \arrow[d, "\mathrm{id}"'] & I_0(v_2) = \underset{v_1}{G} \oplus \underset{e_1}{G} \oplus \underset{e_2}{G} \arrow[d, "\mathrm{Proj}_2"] \\
\mathbb{Z} = \mathbb{1}_{\mathbb{Z}}(e_1) \arrow[r, "\eta_{e_1}"] & I_0(e_1) = G \\
1 \arrow[r, mapsto] & b_2
\end{tikzcd}
$$

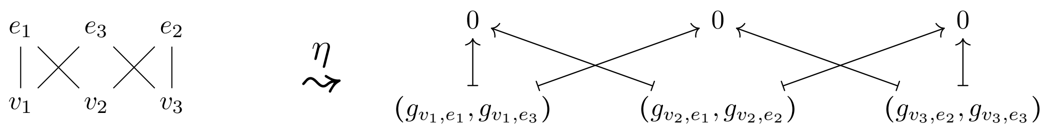
Firstly, we see that choosing an element from $I_0(v_1)$ already determines the choice of the element from $I_0(e_1)$ – this is the case for any simplex and its coface. Secondly, we see that we can not choose the elements from $I_0(v_1)$ and $I_0(v_2)$ independently – from those two particular diagrams, we see that a_2 must be the same as b_2. Altogether, we can independently choose six elements of G, one for each simplex. Schematically, the natural transformation for a choice $(g_{v_1}, g_{v_2}, g_{v_3}, g_{e_1}, g_{e_2}, g_{e_3})$ looks as follows:



Therefore, we have

$$\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_0) = \{\text{maps from } \Sigma \text{ to } G\} \cong G^6.$$

Now let us look at $\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_1)$. Things are a bit easier now, since $I_1(e_j) = 0$ for all the edges, and all the maps in I_1 are 0. This means that we have no dependencies between our choices of elements from $I_1(v_1)$, $I_1(v_2)$, and $I_1(v_3)$. Each of these is $G \oplus G$, so together we can again choose six elements $(g_{v_1,e_1}, g_{v_1,e_3}, g_{v_2,e_1}, g_{v_2,e_2}, g_{v_3,e_2}, g_{v_3,e_3}) \in G^6$:



Hence, we have $\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_1) \cong G^6$.

Lastly, we need to derive the map $\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_0) \to \mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, I_1)$. But this is just post-composition with ∂_0. If we choose elements from groups in I_0, we just map them by the corresponding maps of ∂_0, and get choices of elements from groups in I_1. We get a map $G^6 \to G^6$. Abusing the notation, we denote it by ∂_0. We have

$$
\begin{gathered}
\partial_0 : G^6 \to G^6 \\
\begin{pmatrix} g_{v_1} \\ g_{v_2} \\ g_{v_3} \\ g_{e_1} \\ g_{e_2} \\ g_{e_3} \end{pmatrix} \mapsto \begin{pmatrix} g_{v_1} \\ g_{e_3} - g_{e_1} \\ g_{v_2} \\ g_{e_2} - g_{e_1} \\ g_{v_3} \\ g_{e_3} - g_{e_2} \end{pmatrix},
\end{gathered}
$$

which, abusing the notation a bit more, we can actually express nicely as a coboundary matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Finally, we have

$$R^0\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, \mathbb{1}_G) \cong \ker\partial_0 \cong G,$$

and

$$R^1\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, \mathbb{1}_G) \cong G^6 \big/ \mathrm{im}\,\partial_0 \cong G,$$

which coincides with $H^0(\Sigma; G)$ and $H^1(\Sigma; G)$, respectively.

Exercise 3.7.22. *Is it true that*

$$\begin{aligned} R^n\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, -)(\mathbb{1}_G) &\cong H^n(\Sigma; G) \\ &\quad \big(= R^n\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, \mathbb{1}_G)\big) \end{aligned}$$

The limit functor. The second functor we apply to our resolution is the limit functor

$$\begin{aligned} \varprojlim : \mathrm{Fun}(\Sigma, \mathbf{Ab}) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \varprojlim F \\ \eta : F \Rightarrow G &\longmapsto \varprojlim F \overset{e}{\dashrightarrow} \varprojlim G \end{aligned}$$

Let us first look at the case of just one edge. We get the following diagram:

$$\begin{tikzcd} & \tilde{I}_0(e_1) & \\ \tilde{I}_0(v_1) \arrow[ur] & & \tilde{I}_0(v_2) \arrow[ul] \\ & \varprojlim \tilde{I}_0 \arrow[ul, dashed] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} & G & \\ G \oplus G \oplus G \arrow[ur, "\mathrm{Proj}_2"] & & G \oplus G \oplus G \arrow[ul, "\mathrm{Proj}_2"'] \\ & \varprojlim \tilde{I}_0 \arrow[ul, dashed] \arrow[ur, dashed] & \end{tikzcd}$$

The limit is a pullback of groups

$$\varprojlim \tilde{I}_0 = \big\{(g_1, \dots, g_6) \in G^6 \,\big|\, g_2 = g_4\big\} \cong G^5,$$

where the dotted lines are projections to first three and last three coordinates, respectively.

For the whole triangle, the situation is similar. To obtain the limit, we start with the direct sum $I_0(v_1) \oplus I_0(v_2) \oplus I_0(v_3)$, and take the subgroup on which all the projections commute.

$$\begin{tikzcd} I_0(e_1) & I_0(e_3) & I_0(e_2) \\ I_0(v_1) \arrow[u] \arrow[ur] & I_0(v_2) \arrow[ul] \arrow[ur] & I_0(v_3) \arrow[u] \arrow[ul] \\ & \varprojlim I_0 \arrow[ul, dashed] \arrow[u, dashed] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} G & G & G \\ G \oplus G \oplus G \arrow[u] \arrow[ur] & G \oplus G \oplus G \arrow[ul] \arrow[ur] & G \oplus G \oplus G \arrow[u] \arrow[ul] \\ & \varprojlim I_0 \arrow[ul, dashed] \arrow[u, dashed] \arrow[ur, dashed] & \end{tikzcd}$$

That is, we get

$$\varprojlim I_0 = \big\{(g_1, \dots, g_9) \in G^9 \,\big|\, g_2 = g_8, g_3 = g_5, g_4 = g_9\big\} \cong G^6.$$

For I_1 the situation is even simpler, as we have just zeros on the edges, so there are no commutativity constrains in the diagram:

$$\begin{tikzcd}
I_1(e_1) & & I_1(e_3) & & I_1(e_2) \\
I_1(v_1) \arrow[u] \arrow[urr] & & I_1(v_2) \arrow[ull] \arrow[urr] & & I_1(v_3) \arrow[u] \arrow[ull] \\
& & \varprojlim I_1 \arrow[ull, dashed] \arrow[u, dashed] \arrow[urr, dashed] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 & & 0 & & 0 \\
G \oplus G \arrow[u] \arrow[urr] & & G \oplus G \arrow[ull] \arrow[urr] & & G \oplus G \arrow[u] \arrow[ull] \\
& & \varprojlim I_1 \arrow[ull, dashed] \arrow[u, dashed] \arrow[urr, dashed] & &
\end{tikzcd}$$

The limit ends up being just the product

$$\varprojlim I_1 = I_1(v_1) \oplus I_1(v_2) \oplus I_1(v_3) \cong G^6.$$

The morphism $\varprojlim \partial_0$ is the unique map making the following diagram commute:

$$\begin{tikzcd}
& I_0(v_1) \arrow[r, "(\partial_0)_{v_1}"] & I_1(v_1) & \\
& I_0(v_2) \arrow[r, "(\partial_0)_{v_2}"] & I_1(v_2) & \\
& I_0(v_3) \arrow[r, "(\partial_0)_{v_3}"] & I_1(v_3) & \\
\varprojlim I_0 \arrow[uuur] \arrow[uur] \arrow[ur] \arrow[rrr, dashed, "\varprojlim \partial_0"] & & & \varprojlim I_1 \arrow[uuul] \arrow[uul] \arrow[ul]
\end{tikzcd}$$

But elements in $\varprojlim I_0$ and $\varprojlim I_1$ are exactly the choices we could make for the elements of I_0-groups and I_1-groups in the previous Hom-functor example. And the map induced by ∂_0 acts exactly the same[17]. Therefore, we get

$$R^0 \varprojlim(\mathbb{1}_G) \cong G,$$
$$R^1 \varprojlim(\mathbb{1}_G) \cong G,$$

as before.

Exercise 3.7.23. *Is it true that*

$$R^n \varprojlim(\mathbb{1}_G) \cong H^n(\Sigma; G)?$$

Projective resolutions in $\mathrm{Fun}(\Sigma, \mathbf{Ab})$**?**

People mostly stick with cohomology rather than homology. The reason is that while $\mathrm{Fun}(\Sigma, \mathbf{Ab})$ always has injective resolutions, it might not have projective resolutions (we might need to look toward relatively pathological topological objects without simplicial/CW structure to get a counterexample). We can get a homological-type construction for a simplicial complex by taking a projective resolution of $\mathbb{1}_{\mathbb{Z}}$, and then computing the right derived functor of $- \otimes_{\mathbb{Z}} G$ (known in algebraic contexts as Tor).

[17]This is a more general phenomenon in $\mathbf{Ab}$. As $\mathrm{Fun}(\Sigma, \mathbf{Ab})$ is an abelian category, the set $\mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, F)$ has a structure of abelian group for any functor F, and, therefore, can be seen as an object of $\mathbf{Ab}$. Given any diagram $D : I \to \mathbf{Ab}$, it is not difficult to see that $\varprojlim D \cong \mathrm{Hom}_{\mathrm{Fun}}(\mathbb{1}_{\mathbb{Z}}, D)$.

4 Sheaf Theory

Lecture 8 & 9: Pre(co)sheaves and (co)sheaves

In this lecture we present several definitions for sheaves and cosheaves and show their equivalence under certain assumptions. Each definition has a corresponding problem instance in which it is the most effective formulation. However, we begin by defining presheaves and precosheaves.

Definition 4.9.1. A *presheaf* on X with values in a category $\mathcal{C}$ is a contravariant functor $F : \mathbf{Open}(X) \to \mathcal{C}$.

Definition 4.9.2. A *precosheaf* on X with values in a category $\mathcal{C}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathcal{C}$.

Examples of presheaves and precosheaves.

1. Consider the functor taking open sets on X to functions into some group, in our case $\mathbb{R}$ (we could additionally consider continuous, constant, e.t.c). This is a group with addition obtained by simply adding together the function outputs.

$$\begin{aligned} F : \mathbf{Open}(X) &\to \mathbf{Ab} \\ U &\mapsto \{f : U \to \mathbb{R}\} \end{aligned}$$

   Now consider a morphism in $\mathbf{Open}(X)$ (i.e. $U \subseteq V$)

$$F(U \to V) : F(V) \to F(U),$$

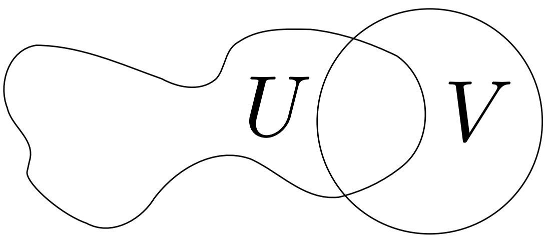
   here $F(U \to V)(f)$ is simply the restriction of f from V to U. Note that the morphism's direction is flipped so F is contravariant and also a presheaf.

2. We can also consider homology as a presheaf.

$$\begin{aligned} \hat{F} : \mathbf{Open}(X) &\to \mathbf{Ab} \\ U &\mapsto H_0(U;\mathbb{Z}) \\ (U \to V) &\mapsto (H_0(U;Z) \to H_0(V;Z)) \end{aligned}$$

   This time the functor is covariant so homology is a precosheaf.

3. Let X be the simple topological space depicted below, with $\mathbf{Open}(X)$, the its corresponding poset category also depicted below. Now let us define the contravariant functor F as



$$\begin{tikzcd} & U \cup V = X & \\ U \arrow[ur] & & V \arrow[ul] \\ & U \cap V \arrow[ul] \arrow[ur] \arrow[uu] & \end{tikzcd}$$

   below. Each open set is mapped to a one dimensional vector space k and the whole space X is mapped to an arbitrary vector space $\mathcal{V}$. The diagram below show the effect of F on the morphisms of $\mathbf{Open}(X)$, where φ is some linear functional.

   Then it is clear F forms a presheaf on X with values in $\mathcal{V}$. Furthermore, if $F(U), F(V)$ and $F(U \cap V)$ are not sufficient to completely determine $F(X)$ then F is just a presheaf and not also a sheaf.

We now turn our attention to defining sheaves and cosheaves distinctly from presheaves and precosheaves.

$$\begin{aligned} F : \mathbf{Open}(X) &\to \mathbf{Vect} \\ U &\mapsto k \\ V &\mapsto k \\ U \cap V &\mapsto k \\ X &\mapsto \mathcal{V} \end{aligned}$$

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\varphi"'] \arrow[dd, "\varphi"] \arrow[dr, "\varphi"] & \\ k \arrow[dr, "\mathrm{id}"'] & & k \arrow[dl, "\mathrm{id}"] \\ & k & \end{tikzcd}$$

Definition 4.9.3. Let $\{U_i\}_{i\in I} = \mathcal{U} \subseteq \mathbf{Open}(X)$ be a a collection of open sets. The *nerve*, $\mathcal{N}(\mathcal{U}) \coloneqq \{J \text{ finite subset of } I : \cap_{j\in J} \neq \varnothing\}$ is a poset category, such that $J \to K$ if $J \subset K$ for $J, K \in \mathcal{N}(\mathcal{U})$

There is a natural functor:

$$i_{\mathcal{U}} : \mathcal{N}(\mathcal{U})^{\mathbf{op}} \longrightarrow \mathbf{Open}(X)$$

$$\begin{tikzcd} J \arrow[r, mapsto] & \bigcap_J U_j \\ K \arrow[u] \arrow[r, mapsto] & \bigcap_K U_k \arrow[u] \end{tikzcd} \quad \text{, if } J \subset K,$$

or alternatively,

$$i_{\mathcal{U}} : \mathcal{N}(\mathcal{U}) \longrightarrow \mathbf{Open}(X)^{\mathbf{op}}$$

$$\begin{tikzcd} J \arrow[r, mapsto] \arrow[d] & \bigcap_J U_j \arrow[d] \\ K \arrow[r, mapsto] & \bigcap_K U_k \end{tikzcd} \quad \text{, if } J \subset K.$$

Here one of the categories, $\mathcal{N}(\mathcal{U})$ or $\mathbf{Open}(X)$ must be flipped to its opposite category so that the functor remains covariant.

Exercise 4.9.4.

1. *Show that* $\varprojlim i_{\mathcal{U}}^{\mathbf{op}} = \cup_{i\in I} U_i$.

2. *Show that* $\varinjlim i_{\mathcal{U}} = \cup_{i\in I} U_i$.

Definition 4.9.5. A *sheaf* F on X with values in a complete category $\mathcal{C}$ is a contravariant functor $F : \mathbf{Open}(X) \to \mathcal{C}$ such that for each collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$ the canonical map $F(\varprojlim i_{\mathcal{U}}^{\mathbf{op}}) \to \varprojlim(F \circ i_{\mathcal{U}}^{\mathbf{op}})$ is an isomorphism.

From Exercise 4.9.4 we know that this limit is just the union, so we can think about sheaves as functors taking unions to limits.

Definition 4.9.6. A *cosheaf* $\hat{F}$ on X with values in a cocomplete category $\mathcal{C}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathcal{C}$ such that for each collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$ the canonical map $\hat{F}(\varinjlim i_{\mathcal{U}}) \to \varinjlim(\hat{F} \circ i_{\mathcal{U}})$ is an isomorphism.

Similarly to sheaves a cosheaf then can be thought of as a functor which commutes with colimits or which takes unions to colimits.

Definition 4.9.7. A *sheaf* F on X with values in a complete category $\mathcal{C}$ is a contravariant functor $F : \mathbf{Open}(X) \to \mathcal{C}$ such that for each collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$ which is stable under finite intersections (i.e. $U_i \cap U_j \in \mathcal{U}$ for all $i, j \in I$), the canonical map $\varprojlim_{\mathcal{U}} F \leftarrow F(\cup_i U_i)$, is an isomorphism.

Here it is beneficial to consider $\mathcal{U}$ as a "subcategory" of $\mathbf{Open}(X)$ (see the diagram below) and F as a functor on this subcategory, $F_{\mathcal{U}} : \mathcal{U} \to \mathcal{C}$ such that $\varprojlim_{\mathcal{U}} F = \varprojlim F_{\mathcal{U}}$.

$$\begin{tikzcd}
U_1 & & U_2 & & U_3 \\
& U_1 \cap U_2 \arrow[ul] \arrow[ur] & & U_2 \cap U_3 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Definition 4.9.8. A *cosheaf* $\hat{F}$ on X with values in a in a complete category $\mathcal{C}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathcal{C}$ such that for each collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$ which is stable under finite intersections, the canonical map $\varinjlim_{\mathcal{U}} \hat{F} \to \hat{F}(\cup_i U_i)$, is an isomorphism.

Note: The following sheaf and cosheaf definitions are valid for any categories with an "element" operation, not just $\mathbf{Set}$ (e.g. $\mathbf{Ab}, \mathbf{Vect}$).

Definition 4.9.9. A *sheaf* F on X with values in $\mathbf{Set}$ is a contravariant functor $F : \mathbf{Open}(X) \to \mathbf{Set}$ such that if $\{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$, $s_i \in F(U_i)$, and $F(U_i \cap U_j \to U_i)(s_i) = F(U_i \cap U_j \to U_j)(s_j)$ for all $i, j \in I$, then there exists a unique $s \in F(\cup_i U_i)$ such that $F(\cup_i U_i \to U_j)(s) = s_j$ for all j.

This definition says if pieces of information from open sets U_i, U_j agree on pairwise intersections then there exists a unique element in the functor applied to the union of all open sets such that restricting to any open sets gives back the initial information. Or more intuitively, if local sets are glued consistently we can recover the initial data from a unique piece of information taken globally.

Definition 4.9.10. A *cosheaf* $\hat{F}$ on X with values in $\mathbf{Set}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathbf{Set}$ such that for each collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$, if $s_{i,j} \in \hat{F}(U_i \cap U_j)$ $(i < j)$, and $\hat{F}(U_i \cap U_j \to U_i)(s_{i,j}) = \hat{F}(U_i \cap U_j \to U_k)(s_{i,k})$ for all $j, k \in I$, then there exists a unique $s \in \hat{F}(\cup_i U_i)$ such that $\hat{F}(U_i \cap U_j \to \cup_i U_i)(s_{i,j}) = s$ for all $i < j$.

Definition 4.9.11. A *sheaf* F on X with values in $\mathbf{Ab}$ is a contravariant functor $F : \mathbf{Open}(X) \to \mathbf{Ab}$ such that for any collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$, the sequence

$$0 \to F(\cup_i U_i) \to \prod_i F(U_i) \xrightarrow{d_0} \prod_{i<j} F(U_i \cap U_j),$$

where

$$d_0((0, \ldots, 0, s_i, 0, \ldots)) = \sum_{i<j} F(U_i \cap U_j \to U_i)(s_i) - \sum_{k<i} F(U_i \cap U_k \to U_i)(s_i),$$

is exact.

Note: The degree map d_0 is exactly the degree map form Čech cohomology.

Definition 4.9.12. A *cosheaf* F on X with values in $\mathbf{Ab}$ is a covariant functor $\hat{F} : \mathbf{Open}(X) \to \mathbf{Ab}$ such that for any collection of open sets $\mathcal{U} = \{U_i\}_{i\in I} \subseteq \mathbf{Open}(X)$, the sequence

$$\oplus_{i<j} \hat{F}(U_i \cap U_j) \xrightarrow{\partial_1} \oplus_i \hat{F}(U_i) \to \hat{F}(\cup_i U_i) \to 0,$$

where

$$\partial_1((0, \ldots, 0, s_{i,j}, 0, \ldots)) = \hat{F}(U_i \cap U_j \to U_i)(s_{i,j}) - \hat{F}(U_i \cap U_k \to U_i)(s_{i,j}),$$

is exact.

Proposition 4.9.13. *Given a presheaf F on X with values in* **Ab** *and a collection $\{U_i\}_{i\in I} = \mathcal{U} \subseteq \mathbf{Open}(X)$, the following are equivalent:*

1. $\varprojlim_{\mathcal{N}(\mathcal{U})} F \xleftarrow{\sim} F(\cup_i U_i)$

2. *If $\exists s_i \in F(U_i)$ s.t. $F(U_i \cap U_j \to U_i)(s_i) = F(U_i \cap U_j \to U_j)(s_j)$ then $\exists! s \in F(\cup U_i)$ s.t. $F(U_j \to \cup U_i)(s) = s_j$*

3. $0 \to F(\cup_i U_i) \xrightarrow{\varepsilon} \prod_i F(U_i) \xrightarrow{d_0} \prod_{i<j} F(U_i \cap U_j)$ *is exact.*

Proof. (2) $\implies$ (3)
Suppose $(s_1, \ldots) \in \ker d_0$, then for every pair of elements the difference,

$$F(U_i \cap U_j \to U_i)(s_i) - F(U_i \cap U_j \to U_j)(s_j) = 0, \ \forall i < j,$$

and hence they must be equal.
Now from (2) we know there exists a unique global element $s \in F(\cup_i U_i)$, that restricts to each local element, or explicitly,

$$F(U_j \to U_i)(s) = s_j, \ \forall j.$$

So s must be the preimage of each element $(s_1, s_2, \ldots) = \varepsilon(s)$. Which is to say, $\ker d_0 \subseteq \operatorname{im} \varepsilon$.

Now since F is a functor it must preserve compositions of morphisms, in particular, $F((U_j \to U_i) \circ (U_k \cap U_j \to U_j)) = F(U_k \cap U_k \to U_i)$. Which implies $\operatorname{im} \varepsilon \subseteq \ker d_0$.

Finally since s was a unique preimage, the map ε must have bee injective. So we conclude the sequence is exact.

(3) $\implies$ (1)
We begin with the diagram from implied by Definition 4.9.5, below left, where we have restriction maps $\varepsilon_k : F(\cup_i U_i) \to F(U_k)$ and definitive limit object as we are in a complete category. Our goal is to show that the unique map to the limit is in fact an isomorphism. By rearranging the diagram and mapping into products rather than distinct open sets, as in below right, we obtain the sequence from (3) which we assume to be exact. Since the sequence is exact, ε is

$$\begin{tikzcd}
F(U_k) \arrow[rr] & & F(U_k \cap U_j) \\
& \varprojlim_{\mathcal{N}(\mathcal{U})} F \arrow[ul, "\varphi_k"'] \arrow[ur] & \\
& F(\bigcup_i U_i) \arrow[uul, "\varepsilon_k"] \arrow[uur] \arrow[u, dashed, "\exists!"] &
\end{tikzcd}$$

$$\begin{tikzcd}
F(\cup_i U_i) \arrow[r, dashed, "\lambda"] \arrow[rr, hook, bend left, "\varepsilon"] & \varprojlim F \arrow[r, "\varphi"] & \prod_k F(U_k) \arrow[r, "d_0"] & \prod_{k<j} F(U_k \cap U_j)
\end{tikzcd}$$

injective and therefore so is λ. Then by the commutivity of the left triangle we know $\operatorname{im} \varepsilon \subseteq \operatorname{im} \varphi$ and additionally by the commutivity of the upper triangle we know $\operatorname{im} \varphi \subseteq \ker d_0$. Combining this with the exactness of the sequence we obtain:

$$\operatorname{im} \varepsilon \subseteq \operatorname{im} \varphi \subseteq \ker d_0 = \operatorname{im} \varepsilon.$$

Now suppose φ were not injective, then it would be possible to define multiple λ maps maintaining the continuity of the diagram, one for each element mapped to the same element by φ, however this violates the uniqueness of λ so φ must be injective. Finally commutivity of the triangle and injectivity of φ implies λ must also be surjective and therefore an isomorphism.

(1) $\implies$ (2)
We want to show if $\exists s_i \in F(U_i)$ s.t. $F(U_i \to U_i \cap U_j)(s_i) = F(U_j \to U_i \cap U_j)(s_j)$ then $\exists! s \in F(\cup U_i)$ s.t. $F(U_j \to \cup U_i)(s) = s_j$. By (1) we have equivalence of the union and the limit then by the definition of the limit we can complete the commutative diagram below with the free group generated by an element $(s_1, s_2, \dots) \in \prod_i F(U_i)$. Since the map from the free group to the limit

$$\begin{tikzcd}
F(U_k) \arrow[rr] & & F(U_k \cap U_j) \\
& \varprojlim F \arrow[ul] \arrow[ur] & \\
& \mathbb{Z}\langle(s_1, s_2, \dots)\rangle \arrow[uul] \arrow[uur] \arrow[u, dashed, "\exists!"] &
\end{tikzcd}$$

$$\begin{tikzcd}
s_k \arrow[rr, mapsto] & & F(U_k \cap U_j \to U_k)(s_k) = F(U_k \cap U_j \to U_j)(s_j) \\
s = \lambda(s_1, \dots) \arrow[u, mapsto] \arrow[urr, mapsto] & & \\
a \cdot (s_1, s_2, \dots) \arrow[uu, bend left, mapsto] \arrow[u, mapsto] \arrow[uurr, mapsto] & &
\end{tikzcd}$$

is unique there must be a unique $s \in \varprojlim F$. Now by the commutivity of the diagram when we restrict $F(U_j \to \cup_i U_i)(s)$ we get s_j as required. □

Proposition 4.9.14. *Suppose F is a sheaf on X with values in* **Ab**. *Then $F(\varnothing) = \langle | \rangle = 0$.*

Proof. Let $\mathcal{U} = \varnothing$. Then

$$0 \to F(\varnothing) \to \prod_{\varnothing} \to \prod_{\varnothing},$$

is exact. But how do we make sense of this sequence? $\prod_{\varnothing} \in \mathbf{Ab}$ is an object such that for each object $G \in \mathrm{Ob}(\mathbf{Ab})$, there is a unique map:

$$\begin{tikzcd}
\prod_{\varnothing} \\
G \arrow[u, dashed, "!"]
\end{tikzcd}$$

There is only one possible map satisfying this diagram for all abelian groups, namely the zero map. So $\prod_{\varnothing} = 0$ and $F(\varnothing) = 0$. □

Exercise 4.9.15.

1. *Suppose F is a sheaf on X with values in* **Set**. *Show that $F(\varnothing)$ is a set with one element.*
2. *What is $\hat{F}(\varnothing)$ for a cosheaf on X with values in* **Ab** *and* **Set***?*

Examples of sheaves and cosheaves.

1. Let $X = [0,1] \subset \mathbb{R}$. Define a functor,

$$F : \mathbf{Open}(X) \to \mathbf{Ab}$$
$$\varnothing \mapsto 0$$
$$U \mapsto \mathbb{Z}$$
$$(U \to V) \mapsto (F(U) \xleftarrow{\text{id}} F(V))$$

Let $U = [0, 1/4]$ and $V = [3/4, 1]$. So by applying F we obtain the diagram:

$$\begin{tikzcd}
F(U) = \mathbb{Z} \arrow[dr, "0"] & & F(V) = \mathbb{Z} \arrow[dl, "0"'] \\
& F(U \cap V) = 0 &
\end{tikzcd}$$

This is exactly the pushout diagram where the restriction maps are 0. So we are looking at pairs of integers which agree under the zero map, i.e. any pair. So then $\varprojlim_{\mathcal{N}(U)} F = \mathbb{Z} \oplus \mathbb{Z} \neq \mathbb{Z}$. Therefore F is merely a presheaf and not a sheaf.

2. Suppose $f : E \to B$ is a continuous map (inspired by vector bundles). The presheaf on B with values in **Set** defined by:

$$F(\mathcal{U}) = \{s : U \to E | f \circ s = \mathrm{id}_{\mathcal{U}}\},$$

is a sheaf. This sheaf is called the *sheaf of sections* of f.

3. Suppose $f : E \to B$ is a continuous map (think vector bundle). The precosheaf on B with values in **Set** defined by

$$\hat{F}(\mathcal{U}) = \pi_0(f^{-1}(\mathcal{U})) = \text{set of path connected components of } f^{-1}(\mathcal{U})$$

is a cosheaf. This cosheaf is called the *Reeb sheaf* of f.

5 Sheaf Cohomology

Lecture 10: Morphisms of sheaves, stalk, sheafification

The aim of this lecture is to show that the category of sheaves $\mathbf{Sh}(X)$ is an abelian category although this is not obvious. Once it is an abelian category we can apply homological algebra (see Section 3) to it.

References. [Cur14, Chapter 2 §5] and the lecture notes of the class taught by Pramod Achar `https://www.math.lsu.edu/~pramod/tc/07s-7280/`[18]

Morphisms of Sheaves

In the following, the paths between sheaves and cosheaves diverge. Therefore the theorems about sheaves and cosheaves become less and less analoguous. We therefore decide for the more straight-forward of the two, sheaves, and discuss only them. Unless stated otherwise, sheaves will have values in **Ab**[19].

To turn sheaves into a category, we need morphisms between sheaves.

Definition 5.10.1 (Morphism of Sheaves)**.** A *morphism* between sheaves on X with values in **Ab** is a natural transformation between the sheaves viewed as functors. In other words, a morphism $\varphi : F \to G$ of sheaves on X is a collection of group homomorphisms $\{\varphi_U \,|\, U \in \mathbf{Open}(X)\}$ such that the diagram

$$\begin{tikzcd}
F(U) \arrow[r, "\varphi_U"] \arrow[d] & G(U) \arrow[d] \\
F(V) \arrow[r, "\varphi_V"] & G(V)
\end{tikzcd}$$

commutes for each pair $V \subseteq U$

[18]The notes by Pramod Achar cover more material than we will here. However, the first 2–3 lecture notes cover similar topics as we do.

[19]Some of the statements would be generalizable to settings without abelian group structure, but for the sake of easyness, we nevertheless restrict ourselves to abelian groups.

Let $\mathbf{Sh}(X)$ be the category of sheaves on X with values in $\mathbf{Ab}$. Definition 5.10.1 can be alternatively phrased using the forgetful functor from sheaves to presheaves

$$\mathrm{For} : \mathbf{Sh}(X) \to \mathrm{Fun}(\mathbf{Open}^{\mathrm{op}}(X), \mathbf{Ab})$$

by defining

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(F, G) := \mathrm{Hom}_{\mathrm{Fun}(\mathbf{Open}^{\mathrm{op}}(X),\mathbf{Ab})}(\mathrm{For}\, F, \mathrm{For}\, G).$$

Kernels and Images of Morphisms of Sheaves: the naïve approach

To define sheaf cohomology using homological algebra (see Section 3) we need the category of sheaves to be abelian. The 3rd example of abelian categories (see Definition 3.6.8) shows that the category of presheaves $\mathrm{Fun}(\mathbf{Open}^{\mathrm{op}}(X), \mathbf{Ab})$ is abelian. However, this does not guarantee that the subcategory of sheaves is abelian as well. We therefore need to check that kernels and cokernels (and hence images and coimages) exist in $\mathbf{Sh}(X)$. Let $\varphi : F \to G$ be a morphism between sheaves, i.e. a natural transformation. The naïve approach fails to provide a definition of the image of a morphism φ between sheaves. However, the naïve approach does work for defining kernels in the category of sheaves. To begin, we recall the definition of kernels and images in the category of presheaves:

$\ker \varphi$ is the contravariant functor mapping an open set U to the abelian group

$(\ker \varphi)(U) := \ker \varphi_U = \ker(F(U) \xrightarrow{\varphi_U} G(U))$

and mapping a morphism $U \subseteq V$ to the induced (see Exercise 5.10.2) group homomorphism $(\ker \varphi)(U) \dashrightarrow (\ker \varphi)(V)$

$\mathrm{im}\, \varphi$ is the contravariant functor mapping an open set U to the abelian group

$(\mathrm{im}\, \varphi)(U) := \mathrm{im}\, \varphi_U = \mathrm{im}(F(U) \xrightarrow{\varphi_U} G(U))$

and mapping a morphism $U \subseteq V$ to the induced (see Exercise 5.10.2) group homomorphism $(\mathrm{im}\, \varphi)(U) \dashrightarrow (\mathrm{im}\, \varphi)(V)$

Exercise 5.10.2. *Show that for $U \subseteq V$ the definition of $(\ker \varphi)(V)$ as the kernel of φ_V indeed induces a unique morphism $(\ker \varphi)(U) \dashrightarrow (\ker \varphi)(V)$, and that the definition of $(\mathrm{im}\, \varphi)(V)$ as the kernel of the cokernel-map of φ_V indeed induces a unique morphism $(\mathrm{im}\, \varphi)(U) \dashrightarrow (\mathrm{im}\, \varphi)(V)$ with the help of the induced map between the cokernels.*

If $\ker \varphi$ and $\mathrm{im}\, \varphi$ fulfilled the sheaf condition of Definition 4.9.11 we could use these two definitions to show existence of kernels and images in the category of sheaves.

Proposition 5.10.3. *Let $\varphi : F \to G$ be a morphism of sheaves. Then $\ker \varphi$ is a sheaf.*

Proof. A diagram chasing proof (left to the reader) in the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\ker \varphi)(\bigcup_i U_i) \arrow[r, "\alpha"] \arrow[d] & \prod_i (\ker \varphi)(U_i) \arrow[r, "\beta"] \arrow[d] & \prod_{i<j} (\ker \varphi)(U_i \cap U_j) \arrow[d] \\
0 \arrow[r] & F(\bigcup_i U_i) \arrow[r] \arrow[d] & \prod_i F(U_i) \arrow[r] \arrow[d] & \prod_{i<j} F(U_i \cap U_j) \arrow[d] \\
0 \arrow[r] & G(\bigcup_i U_i) \arrow[r] & \prod_i G(U_i) \arrow[r] & \prod_{i<j} G(U_i \cap U_j)
\end{tikzcd}$$

shows that $\ker \varphi$ fulfills the sheaf condition of Definition 4.9.11 and therefore is a sheaf. □

The above proposition fails for $\mathrm{im}\, \varphi$. Indeed $\mathrm{im}\, \varphi$ does not necessarily need to be a sheaf (see Section 6.2 for an example). There is still hope, however, that images of morphisms exist in $\mathbf{Sh}(X)$. We only need to define the image of a morphism differently. For this, we try to find out what went wrong with $\mathrm{im}\, \varphi$: How can a presheaf fail to be a sheaf?

To answer this question we use the $\mathbf{Set}$-definition of sheaves (Definition 4.9.9). The statement "then there exists a unique $s \in F(\cup_i U_i)$" from this definition can either fail to be true by lacking existence ("not enough global information") or by lacking uniqueness ("too much global information").

Example 5.10.4 (Non-existence of sections)**.** For $X = U \sqcup T$ and $\mathbf{Open}(X) = \{\varnothing, U, T, X\}$, consider the presheaf F mapping U, T, X to a vector space V and identity maps between them and mapping $\varnothing$ to 0. This presheaf fails to be a sheaf, because for $w \in V = F(U)$ and $z \in V = F(T)$ with $w \neq z$, there exists no $v \in V = F(X)$ that maps both to w and to z, although w and z both map to $0 \in 0 = F(\varnothing)$. In other words, $F(X)$ is not large enough.

Example 5.10.5 (Non-uniqueness of sections)**.** For $X = U \cup T$ with $U \cap T \neq \varnothing$ and $\mathbf{Open}(X) = \{\varnothing, U \cap T, U, T, X\}$, consider the presheaf F mapping $U \cap T, U, T$ to a field (and thus a vector space) k and identity maps between them, mapping X to a larger vector space $V \neq k$ over k with a linear map $\varphi : V \to k$ such that $F(U \subset X) = F(T \subset X) = F(U \cap T \subset X) = \varphi$. This presheaf fails to be a sheaf, because for $0 \in k = F(U \cap T) = F(U) = F(T)$ there is more than one $v \in \ker \varphi \subset F(X)$ with $\varphi(v) = 0$ and therefore more than one section over X which restricts to 0 over U, T, and $U \cap T$. In other words, $F(X)$ is too large.

Both these problems are global problems, not local problems. This motivates the procedure of sheafification: taking a presheaf that is not a sheaf, extracting the local information and composing the local information smartly to form exactly the right amount of global information, yielding a sheaf. We will then define the image of φ as the sheafification of the above defined presheaf $\operatorname{im} \varphi$.

Sheafification

The local information of a presheaf is its stalk:

Definition 5.10.6 (Stalk)**.** The *stalk* of presheaf F (with values in a category $\mathcal{C}$ which is complete and cocomplete, like **Set** or **Ab**) at $x \in X$ is

$$F_x \coloneqq \varinjlim_{U \ni x} F(U)$$

where the colimit is taken over all open sets U which contain x.

Example 5.10.7. Suppose X is a metric space. Let $x \in B_{\frac{1}{n}}(x) \subset B_{\frac{1}{n-1}}(x) \subset \cdots \subset B_1(x)$ be the sequence of open balls around x with decreasing radius. Let I be the functor from the poset category $(\mathbb{N}, \leq)$ to $\mathbf{Open}^{\mathrm{op}}(X)$, defined by

$$\begin{tikzcd} I: & \mathbb{N} \arrow[r] & \mathbf{Open}^{\mathrm{op}}(X) \\ & n \arrow[r, mapsto] & B_{\frac{1}{n}}(x) \\ & m \arrow[u] \arrow[r, mapsto] & B_{\frac{1}{m}}(x) \arrow[u] \end{tikzcd}$$

Composing I with the presheaf $F : \mathbf{Open}^{\mathrm{op}}(X) \to \mathbf{Ab}$, yields

$$\begin{aligned} F \circ I : \quad & \mathbb{N} \longrightarrow \mathbf{Ab} \\ & n \longmapsto F\left(B_{\frac{1}{n}}(x)\right) \quad =: G_n \end{aligned}$$

The colimit of this functor, Example 2.4.23, is

$$\varinjlim_{U \ni x} F(U) = \varinjlim F \circ I = \bigoplus_{n \in \mathbb{N}} G_n \big/ \sim$$

where the equivalence relation is given by $(0, \cdots, 0, g_l, 0 \cdots) \sim (0, \cdots, 0, h_k, 0 \cdots)$ if there exists m such that $((F \circ I)(l \leq m))(g_l) = ((F \circ I)(k \leq m))(h_k)$[20].

[20]The equivalence relation here is defined only on a set of basis vectors for $\bigoplus G_n$ and should then be extended linearly to a relation on the entire vector space.

Remark. The idea of Definition 5.10.6 is that F is only defined on open sets and not on the singleton $\{x\}$, we therefore take the colimit over all neighborhoods. The same idea can be applied to any subset $Y \subseteq X$. Consider

$$F_Y := \varinjlim_{U \supseteq Y \text{ open}} F(U)$$

as the 'data' which F assigns to Y.

The following proposition describes how the global information connects with the local information.

Definition 5.10.8. Let F be a presheaf on X with values in **Set**. For $U \in \mathbf{Open}(X)$ and $x \in U$, let

$$F(U) \dashrightarrow F_x$$

be the unique morphism given from the definition of colimits (recall $F_x := \varinjlim_{V \ni x} F(V)$). Define s_x to be the image of $s \in F(U)$ under this map.

Unpacking Definition 5.10.6 for the special case of $\mathcal{C} = \mathbf{Ab}$—generalizing Example 5.10.7—yields the following equivalent definition of the stalk.

Definition 5.10.9 (Stalk in **Ab**)**.** Let F be a presheaf on X with values in **Ab**. The *stalk* of F at $x \in X$, F_x, is the abelian group consisting of equivalence classes of pairs

$$(U, s) : U \ni x \text{ open and } s \in F(U)$$

where $(U, s) \sim (V, t)$ if $\exists W \subseteq U \cap V$ open such that $x \in W$ and $F(W \to U)(s) = F(W \to V)(t)$. The group operation is defined by $(U, s) + (V, t) := (U \cap V, F(U \cap V \to U)(s) + F(U \cap V \to V)(t))$.

Using this definition, the universal map from Definition 5.10.8 is given by

$$\begin{aligned} F(U) &\to F_x \\ s &\mapsto s_x := [(U, s)]_\sim \end{aligned}$$

Also the global information about morphisms connects to the local information. Suppose $\gamma : F \to G$ is a morphism of presheaves (hence a natural transformation). Then γ induces a morphism of stalks for each $x \in X$:

$$\begin{aligned} \gamma_x : F_x &\to G_x \\ [(U, s)]_\sim &\mapsto [(U, \gamma_U(s))]_\sim \end{aligned}$$

where $\gamma_U : F(U) \to G(U)$ is the group homomorphism from the definition of natural transformations.

Exercise 5.10.10. *Show that γ_x is indeed well-defined by showing*

$$(U, s) \sim_F (V, t) \implies (U, \gamma_U(s)) \sim_G (V, \gamma_V(t))$$

Finally, we can compose the local information of a presheaf to form global information.

Definition 5.10.11 (Sheafification)**.** Let F be a presheaf on X with values in **Set**. The *sheafification* of F, denoted F^+, is defined to be

$$F^+(U) = \left\{ s : U \to \bigsqcup_{x \in U} F_x \;\middle|\; \begin{array}{l} \forall x \in U : s(x) \in F_x \text{ and} \\ \forall x \in U \;\exists V \text{ with } x \in V \subseteq U \;\exists t \in F(V) \;\forall y \in V : s(y) = t_y \end{array} \right\}.$$

for each open set $U \subset X$, where $\bigsqcup$ denotes the disjoint union. In other words, s chooses for every point $x \in U$ an element $s(x)$ in its stalk F_x in a locally constant / locally coherent way: in small enough neighborhoods V the chosen stalk element $s(y)$ will always be induced by the same element t.

To turn F^+ into a functor we define F^+ of an inclusion $V \subseteq U$ to be the group homomorphism $\cdot|_V : F^+(U) \to F^+(V)$ that maps a function s to its restriction $s|_V$.

Exercise 5.10.12. *Prove that F^+ is a sheaf for every presheaf F.*

Definition 5.10.11 can be rephrased using the following notation for Cartesian products: Suppose $s \in \prod_{x \in U} F_x$. Let $\pi_x(s)$ denote the projection of s onto F_x.

Definition 5.10.13 (Alternative Definition of Sheafification)**.** Let F be a presheaf on X with values in $\mathbf{Ab}$. The *sheafification* of F, denoted F^+, is defined to be

$$F^+(U) = \left\{ s \in \prod_{x \in U} F_x \;\middle|\; \forall x \in U \; \exists V \text{ with } x \in V \subseteq U \; \exists t \in F(V) \; \forall y \in V : \pi_y(s) = t_y \right\}$$

for each open set $U \subset X$.

Exercise 5.10.14. *Prove that* $(.)^+ : \mathrm{Fun}(\mathbf{Open}^{op}(X), \mathbf{Set}) \to \mathbf{Sh}(X)$ *is a functor. For this, define γ^+ of a morphism (i.e. natural transformation) γ between two presheaves F and G as* $\gamma^+_U : F^+(U) \to G^+(U), s \mapsto \left(\prod_{x \in U} \gamma_x\right)(s)$.

To show that this functor $(.)^+$ is a left adjoint of the forgetful functor (see Theorem 5.10.17), we first need a Lemma.

Lemma 5.10.15. *If F is a sheaf, then $F \simeq (For(F))^+$.*

Proof. Exercise to the reader. □

In other words, sheafification does not change sheaves.

Lemma 5.10.16. *If F is a presheaf, then $\forall x \in X$*

$$F_x \simeq (F^+)_x$$

Proof. Exercise to the reader. □

In other words, sheafification does not change the local information.

Theorem 5.10.17. *The functor* $(.)^+ : \mathrm{Fun}(\mathbf{Open}^{op}(X), \mathbf{Ab}) \to \mathbf{Sh}(X)$ *is a left adjoint of the forgetful functor* $For : \mathbf{Sh}(X) \to \mathrm{Fun}(\mathbf{Open}^{op}(X), \mathbf{Set})$. *I.e.*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(F^+, G) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathbf{Open}^{op}(X), \mathbf{Set})}(F, For(G))$$

is natural.

Proof. The natural bijection is given by $\gamma^+ \leftrightarrow \gamma$. The remainder of the proof can be found in the literature. □

Definition of Images of Morphisms of Sheaves

Finally, we can define the image of a morphism as the sheafification of the attempt $\mathrm{im}(\mathrm{For}(\gamma))$ we tried before, which turned out not to be a sheaf.

Definition 5.10.18 (Image of a Morphism of Sheaves)**.** Let $\gamma : F \to G$ be a morphism of sheaves. Define

$$\mathrm{im}\,\gamma \coloneqq (\mathrm{im}(\mathrm{For}(\gamma)))^+$$

Moreover, one can show that there is a natural inclusion of sheaves: $\mathrm{im}\,\gamma \to G$.

Definition 5.10.19. Let $\gamma : F \to G$ be a morphism of sheaves.

- γ is injective if $\ker \gamma = 0$
- γ is surjective if $\mathrm{im}\,\gamma \cong G$

The fact that the definition of image needs sheafification (contrarily to the definition of kernel which already is a sheaf) is a hint that surjectivity of morphisms between sheaves will be more complicated than injectivity.

Proposition 5.10.20. *γ is injective iff $\gamma(U) : F(U) \to G(U)$ is injective for all $U \in$ **Open**(X).*

Proof. Follows from the definition of ker γ. □

This is not true for surjectivity! It is not enough to check only the open sets, but checking the colimits of the open sets, i.e. the stalks, is enough.

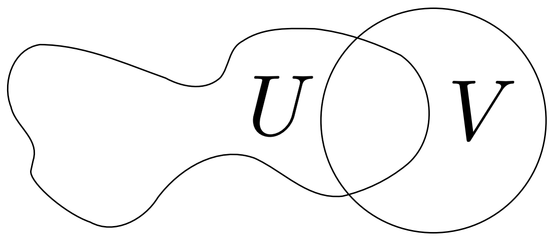
Proposition 5.10.21. *A morphism $\gamma : F \to G$ of sheaves is*

- *surjective iff $\gamma_x : F_x \to G_x$ is surjective $\forall x \in X$*
- *injective iff $\gamma_x : F_x \to G_x$ is injective $\forall x \in X$*
- *isomorphism iff $\gamma_x : F_x \to G_x$ is isomorphism $\forall x \in X$*

Proof. Exercise to the reader. □

Here, it is important that the isomorphisms between the stalks are all induced by the same morphism $\gamma : F \to G$. If each of the stalks F_x was isomorphic to G_x with individual isomorphisms, this would not be enough to conclude that F and G are isomorphic. In other words, for $F, G \in \mathbf{Sh}(X)$, we have $F \simeq G$ in $\mathbf{Sh}(X)$ implies $F_x \simeq G_x$ in $\mathbf{Ab}$, $\forall x \in X$ (see Proposition 5.10.21). But there are counter-examples for the implication in the other direction. We will give such an example now. Another example can be constructed from the sections of the coverings of Example 5.12.43.1 and Example 5.12.43.2 (see Example 5.12.51.3). The intuition to find a counter-example is: The stalks give the local information in each point, but not how the local information of one point relates to (or 'is glued to') the local information of another point.

Example 5.10.22. Suppose $X = U \cup V$ has 5 open sets: $\mathbf{Open}(X) = \{X, U, V, W \coloneqq U \cap V, \varnothing\}$



Define

$$\begin{aligned} F(Z) &= \mathbb{Z} \quad \forall \text{ nonempty } Z \in \mathbf{Open}(X), \\ F(\varnothing) &= 0, \\ \text{and } F(Z_1 \to Z_2) &= (id)_{\mathbb{Z}} \quad \forall \text{ nonempty } Z_i \in \mathbf{Open}(X) \end{aligned}$$

$$\begin{tikzcd} & X = U \cup V & \\ U \arrow[ur] & & V \arrow[ul] \\ & W = U \cap V \arrow[ul] \arrow[ur] \arrow[uu] & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"] & \\ \mathbb{Z} \arrow[dr, "\mathrm{id}"'] & & \mathbb{Z} \arrow[dl, "\mathrm{id}"] \\ & \mathbb{Z} & \end{tikzcd}$$

Exercise 5.10.23. *Show that F is a sheaf, for example by using the* **Set***-definition of sheaves (Definition 4.9.9)*

Next, we compute the stalks. Let $x \in U \setminus W$. Then

$$F_x = \varinjlim_{Z \ni x} F(Z)$$

is the colimit of the finite diagram $F(X) \to F(U)$ which has a terminal object, namely $F(U)$. Therefore

$$F_x = \varinjlim_{Z \ni x} F(Z) = F(U) = \mathbb{Z}.$$

Let $x \in W$, then the stalk F_x is the colimit of the finite diagram $F(X) \to F(U) \to F(W)$ which has $F(W) = \mathbb{Z}$ as its terminal object. Hence, again $F_x = \mathbb{Z}$.

In general, if there are only finitely many open set, the stalk F_x is the functor applied to the smallest open set containing x, which is in this example always $\mathbb{Z}$.

Now we will construct a different sheaf, with the same stalks.

Example 5.10.24. Let X be the same topological space as before. Define

$$G(Z) = \begin{cases} \mathbb{Z} & Z = U, V, \text{ or } W \\ \mathbb{Z} \oplus \mathbb{Z} & Z = X \\ 0 & Z = \varnothing. \end{cases}$$

$$\begin{aligned} G(X \to U) &= \mathrm{proj}_1 \\ G(X \to V) &= \mathrm{proj}_2 \\ G(U \to W) &= 0 \\ G(V \to W) &= 0 \end{aligned}$$

$$\begin{tikzcd} & X = U \cup V & \\ U \arrow[ur] & & V \arrow[ul] \\ & W = U \cap V \arrow[ul] \arrow[ur] \arrow[uu] & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, "\mathrm{proj}_1"'] \arrow[dr, "\mathrm{proj}_2"] \arrow[dd, "0"] & \\ \mathbb{Z} \arrow[dr, "0"'] & & \mathbb{Z} \arrow[dl, "0"] \\ & \mathbb{Z} & \end{tikzcd}$$

Exercise 5.10.25. *Show that G is a sheaf, for example by using the* **Set***-definition of sheaves (Definition 4.9.9)*

Again we compute the stalks by

$$G_x = G(\text{smallest open neighborhood containing } x) = \mathbb{Z} \quad \forall x \in X.$$

And we observe that we get the same stalks as in Example 5.10.22, although the sheaves F and G are not isomorphic (because $F(X)$ and $G(X)$ are not isomorphic).

Lecture 11: Category of sheaves, sheaf cohomology

References. [Cur14, Chapter 2] and [Bre12, Chapter 1 Section 3][21]

As a first example of sheafification we sheafify the constant presheaf, which failed to be a sheaf (see the 1st example after the definitions of sheaves and cosheaves).

Example 5.11.26. Let $I = [0,1] \subset \mathbb{R}$ with the euclidean topology inherited from $\mathbb{R}$, let G be a non-trivial abelian group and

$$\begin{aligned} F : \mathbf{Open}^{\mathrm{op}}(I) &\to \mathbf{Ab} \\ U &\mapsto G \\ (U \to V) &\mapsto \mathrm{id}_G. \end{aligned}$$

[21] Bredon uses Definition 1.0.3 to define sheaves. We have not had time to show that this is equivalent to our definition (see Definitions 4.9.5, 4.9.7, 4.9.9 and 4.9.11), but it is.

F is not a sheaf (because firstly, $F(\varnothing) = G \neq 0$ and after correcting this, two disconnected sets would need to get mapped to two copies of G in order to get the sheaf property (see the 1st example after the definitions of sheaves and cosheaves)). We compute its sheafification F^+. For computing,

$$F^+(U) = \left\{ s : U \to \coprod_{x \in U} F_x \;\middle|\; \begin{array}{l} \forall x \in U : s(x) \in F_x \text{ and} \\ \forall x \in U \ \exists V \text{ with } x \in V \subseteq U \ \exists t \in F(V) \ \forall y \in V : s(y) = t_y \end{array} \right\},$$

we first need to compute the stalks $\forall x \in I$ using Definition 5.10.9:

$$F_x = \{(U,s) \,|\, U \ni x \text{ open and } s \in F(U)\}_{\diagup \sim} \tag{1}$$

$$= \{(U,g) \,|\, U \ni x \text{ open and } g \in G\}_{\diagup \sim} \tag{2}$$

where $(U,g) \sim (V,h)$ iff $\exists W \subseteq U \cap V$ neighborhood of x s.t. $F(U \to W)(g) = F(V \to W)(h)$. As the morphisms $F(U \to W)$ and $F(V \to W)$ are the identity and as $U \cap V$ is a suitable candidate for W, this is equivalent to: $(U,g) \sim (V,h)$ iff $g = h$. This thus yields:

$$F_x \cong G \quad \forall x \in I.$$

Secondly, we need to compute the induced maps $F(U) \to F_x$, see Definition 5.10.8.

$$\begin{array}{ccccc} G = F(U) & \longrightarrow & F_x & \xleftrightarrow{\cong} & G \\ g & \longmapsto & g_x := [(U,g)]_\sim & \xleftrightarrow{\cong} & g, \end{array}$$

yielding again the identity map on G. In other words, through the identification of F_x with G, g_x gets identified with g. We can therefore simplify $F^+(U)$ to

$$\begin{aligned} F^+(U) &= \{s : U \to G \,|\, \forall x \in U \ \exists V \text{ with } x \in V \subseteq U \ \exists t \in F(V) \ \forall y \in V : s(y) = t\} \\ &= \{s : U \to G \,|\, \forall x \in U \ \exists V \text{ with } x \in V \subseteq U : s|_V \text{ is constant}\} \\ &= \{s : U \to G \,|\, s \text{ is locally constant}\}. \end{aligned}$$

We observe

- A set of connected open neighborhoods, $\{V_x\}_{x \in U}$, form an open cover of U.
- If $V_x \cap V_y \neq \varnothing$ then $s(x) = s(y) \in G$,

and infer that $s \in F^+(U)$ is completely determined by a choice of $t \in G$ for each connected component of U. Another way to say this, is

$$F^+(U) = \{s : U \to G \,|\, s \text{ continuous}\},$$

where G has the discrete topology.

We introduced sheafification to define the image of a morphsim because existence of images is necessary for a category to be abelian.

Theorem 5.11.27. *The category of sheaves on X with values in* $\mathbf{Ab}$*,* $\mathbf{Sh}(X)$*, is an abelian category. Moreover, each sheaf $F \in \mathbf{Sh}(X)$ has an injective resolution*[22]

Proof. The proof can be read up in the references. □

This is what we need in order to use homological algebra (see Section 3). Applying homological algebra to sheaves is called sheaf cohomology. The roadmap is the following:

Given a left exact (additive) functor[23]

$$\Lambda : \mathbf{Sh}(X) \to \mathbf{Ab}$$

[22] But surjective resolutions do not need to exist. This is the reason why in the following we will work with right derived functors instead of left derived functors. And this explains why homology and sheaves do not work together well, but cohomology and sheaves do.

[23] In the following, we will denote sheaves with capital letters and functors from $\mathbf{Sh}(X)$ to $\mathbf{Ab}$ with capital Greek letters.

and a sheet $F \in \mathbf{Sh}(X)$, we "can"[24] find an injective resolution of F:

$$0 \to F \to I_0 \to I_1 \to \dots$$

and we "can" compute

$$0 \overset{\partial_0}{\to} \Lambda(I_0) \overset{\partial_1}{\to} \Lambda(I_1) \to \dots \in \mathbf{Ab}$$

and we can define the cohomology of X with coefficients in the sheaf F as the right derived functor of Λ, which is defined as the cohomology of the above sequence, i.e.

$$H^i_\Lambda(X;F) \coloneqq (R^n\Lambda)(F) \coloneqq {}^{\ker \partial_n}\!\big/_{\operatorname{im}\partial_{n-1}} \in \mathbf{Ab}.$$

Next, we explore which left exact additive functors $\Lambda : \mathbf{Sh}(X) \to \mathbf{Ab}$ to use and how to find injective resolutions. One idea for a functor Λ is to use a representable functor $\Lambda \coloneqq \operatorname{Hom}_{\mathbf{Sh}(X)}(F,-)$, represented by a convenient sheaf $F \in \mathbf{Sh}(X)$. A good canditate for F is the sheaf from Example 5.11.26.

Definition 5.11.28 (Constant Sheaf)**.** The *constant sheaf*, $\mathbb{1}_G$, is the sheafification of the functor which maps each open set to the abelian group G and each morphism to id_G. As Example 5.11.26 has shown, this is equivalent to defining:

$$\mathbb{1}_G \coloneqq \{\text{continous functions from } U \text{ to } G\}$$

where G has the discrete topology.

Example 5.11.29.

$$\operatorname{Hom}_{\mathbf{Sh}(X)}(F,-) : \mathbf{Sh}(X) \to \mathbf{Ab}$$

is additive and left exact. Specifically, we will focus on the case where $F = \mathbb{1}_G$:

$$\operatorname{Hom}_{\mathbf{Sh}(X)}(\mathbb{1}_G,-).$$

Another idea for a functor $\Lambda : \mathbf{Sh}(X) \to \mathbf{Ab}$ is the functor that evaluates a given sheaf on the open set consisting of the whole space.

Definition 5.11.30 (Global Section Functor)**.** Given a sheaf $F \in \mathbf{Sh}(X)$, define

$$\Gamma(X;F) \coloneqq F(X) \in \mathbf{Ab}.$$

$\Gamma(X;-) : \mathbf{Sh}(X) \to \mathbf{Ab}$ is called the *global sections functor.*

Exercise 5.11.31. *Prove that* $\Gamma(X;-)$ *is a additive and left exact functor.*

A third idea for getting a functor $\Lambda : \mathbf{Sh}(X) \to \mathbf{Ab}$ is to push-forward a sheaf $F \in \mathbf{Sh}(X)$ to a sheaf in $\mathbf{Sh}(\{p\})$ where the only information the sheaf gives is the abelian group to which it is evaluated for the set $\{p\}$. We therefore define push-forwards and pull-backs.

Definition 5.11.32 (push-forward)**.** Suppose $f : X \to Y$ is a continuous map.
If F is a sheaf on X, define the *direct image of F along f* (or *push-forward sheaf*) by

$$\begin{aligned} f_*(F) \coloneqq{}& \text{the sheaf which maps} \\ & U \subseteq Y \mapsto F(f^{-1}(U)) \end{aligned}$$

Similarly, we would like to define the pull-back sheaf of a sheaf $G \in \mathbf{Sh}(X)$ along f as mapping an open set $U \subseteq X$ to $G(f(U))$. However, $f(U)$ does not need to be open. Therefore, we need to apply the same trick as in the definition of stalk, namely using the colimit over all open sets containing $f(U)$ instead. Furthermore, the thus defined presheaf is not a sheaf. Hence we need to sheafify it:

[24] An injective resolution needs to exist but usually one doesn't know how to find it.

Definition 5.11.33 (pull-back)**.** Suppose $f : X \to Y$ is a continuous map.
If G is a sheaf on Y, define the *inverse image of G along f* (or *pull-back sheaf*) by

$$\begin{aligned} f^*(G) &:= \text{the sheafification of the presheaf which maps} \\ &\qquad U \subseteq X \mapsto \varinjlim_{V \supseteq f(U)} G(V) \end{aligned}$$

Example 5.11.34. Let $f : X \to \{p\}$, where $\{p\}$ denotes the topological space consisting of only one point. Let G be an abelian group. Let $\mathbb{1}$ be the sheaf in $\mathbf{Sh}(\{p\})$ with $\mathbb{1}(\{p\}) = G$ and $\mathbb{1}(\varnothing) = 0$. Then $f^*\mathbb{1} = \mathbb{1}_G$, yielding yet another equivalent way to define the constant sheaf, see Definition 5.11.28.

Theorem 5.11.35. *Let $f : X \to Y$ be a continuous map. Then*

$$\begin{aligned} f_* &: \mathbf{Sh}(X) \to \mathbf{Sh}(Y) \text{ and} \\ f^* &: \mathbf{Sh}(Y) \to \mathbf{Sh}(X) \end{aligned}$$

are functors. Moreover,

- *f_* is left exact,*
- *f^* is exact,*
- *(f^*, f_*) form an adjoint pair.*

Proof. The proof can be found in the references. □

Adjoint pairs can be used to translate problems from complicated categories to easier categories. The adjoint pair (f^*, f_*) can translate problems from $\mathbf{Sh}(X)$, X being a large topological space, to $\mathbf{Sh}(Y)$, Y being a small topological space. In the following, we will do this for Y being the topological space, $\{\{p\}\}$.

Example 5.11.36. Let $f : X \to \{p\}$ be the constant map. We can push-forward a sheaf on X to a sheaf on $\{p\}$. The category $\mathbf{Sh}(\{p\})$ is equivalent to the category $\mathbf{Ab}$ because the only information that a sheaf on $\{p\}$ caries is the abelian group that is its global section. We therefore get a functor from $\mathbf{Sh}(X)$ to $\mathbf{Ab}$ by composing

$$\mathbf{Sh}(X) \xrightarrow{f_*} \mathbf{Sh}(\{p\}) \xrightarrow{\Gamma(\{p\};-)} \mathbf{Ab}.$$

Exercise 5.11.37. $\Gamma(X;-) = \Gamma(\{p\}; f_*(-))$ *as functors from* $\mathbf{Sh}(X)$ *to* $\mathbf{Ab}$. *Moreover,* $\Gamma(\{p\}; f_*(-))$ *is additive and left exact.*

As the functors are equal, their right derived functors are equal:

$$\left(R^i\Gamma(X;-)\right)(F) = \left(R^i\Gamma(\{p\}; f_*(-))\right)(F).$$

Notation: As a shorthand for $\Gamma(\{p\}; f_*(F)) = (f_*(F))(\{p\})$ one can write $f * (F)$ because sheaves on $\{p\}$ can only be non-trivially evaluated on $\{p\}$. Thus, as a shorthand for $\Gamma(\{p\}; f_*(-))$ one can write f_*, yielding the shorter statement:

$$\left(R^i\Gamma(X;-)\right)(F) = \left(R^i f_*\right)(F).$$

Proposition 5.11.38. *Let $F \in \mathbf{Sh}(X)$. Then*

$$\left(R^i\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathbb{1}_{\mathbb{Z}}, -)\right)(F) \simeq \left(R^i\Gamma(X;-)\right)(F).$$

Proof. Should be checked by the reader. □

Together, this shows that all three ideas for left exact additive functors $\Lambda : \mathbf{Sh}(X) \to \mathbf{Ab}$ yield the same right derived functor, when choosing $G := \mathbb{Z}$, the free abelian group. We can therefore use any of the three for the following definition.

Definition 5.11.39 (Sheaf Cohomology)**.** Define

$$H^i(X;F) := H^i_{\Gamma(X;-)}(X;F) = \left(R^i\Gamma(X;-)\right)(F)$$

The following example shows that sheaf cohomology of the constant sheaf agrees with the traditional singular cohomology with coefficients in a group G, under mild assumptions on X (for example if X is homotopy equivalent to a CW complex).

Example 5.11.40. If X is a manifold (or, more generally, homotopy equivalent to a CW complex), then $H^i(X;\mathbb{1}_G) = H^i(X;G)$.

Recall that $H^i(X;\mathbb{1}_G) = \left(R^i\Gamma(X;-)\right)(\mathbb{1}_G) = \left(R^i\Gamma(\{p\};f_*(-))\right)(\mathbb{1}_G)$ is the right derived functor of pushing forward to a point and taking global sections of the point. Similarly, we can push-forward to the slightly larger space $\mathbb{R}$.

Example 5.11.41. Suppose $f : M \to \mathbb{R}$ is a Morse function (i.e. a function to the real line that is differentialable and has separated critical values). Then, for a sheaf $F \in \mathbf{Sh}(M)$ on M, $f_*(F) \in \mathbf{Sh}(\mathbb{R})$ is a sheaf on $\mathbb{R}$. For $p \in \mathbb{R}$ and $\iota_p : \{p\} \hookrightarrow \mathbb{R}$, $(\iota_p^* \circ f_*)(F) \in \mathbf{Sh}(\{p\})$ is a sheaf on a point and can therefore be represented by its global section $\Gamma(\{p\};(\iota_p^* \circ f_*)(F)) \in \mathbf{Ab}$, in short notation, $(\iota_p^* \circ f_*)(F) \in \mathbf{Ab}$. As the functor $\Gamma(\{p\};(\iota_p^* \circ f_*)$, in short, $\iota_p^* \circ f_*$, is left exact and additive, we can build its right derived functor $R^i\Gamma(\{p\};(\iota_p^* \circ f_*))$, in short $R^i(\iota_p^* \circ f_*)$.

Let $\mathbb{1}_G \in \mathbf{Sh}(M)$ be the constant sheaf on M. Then,

$$\left(R^i(\iota_p^* \circ f_*)\right)(\mathbb{1}_G) \simeq H^i(f^{-1}(\{p\});G) \tag{3}$$

This reminds of (level set) persistent cohomology and hence raises a question: Can we think about (level set) persistent (co-)homology as a sheaf on $\mathbb{R}$? And if so, can this be generalized to $\mathbb{R}^n$ instead of $\mathbb{R}$, yielding new insights on the decomposables of multi-parameter persistence?

Lecture 12: Étale space, local systems, Σ-constructibility

The étalé perspective of sheaf theory:

- Covering spaces and fundamental groups
- Fiber bundles étalé spaces, and constant sheaves
- Local systems and representations of $\pi_1(X;x_0)$

Motivation: We want to compute cohomology, but using the general definitions, we have too many open sets to consider. If we want to actually compute homology, we need to throw away 'redundant' open sets. In order to make this approach concrete, we will take a detour through the theory of covering spaces and the classification of local systems.

5.12.1 Covering spaces and fundamental groups

We review a classical algebraic topology result drawing a relation between covering spaces and the fundamental group. More details can be found, e.g., in Hatcher's Algebraic Topology.

Definition 5.12.42 (Covering space)**.** A *covering space* of a topological space X is a topological space $\tilde{X}$ together with a continuous map

$$p : \tilde{X} \to X,$$

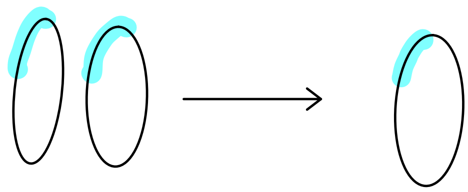
satisfying the following condition: for each $x \in X$ there exists a neighbourhood U of x, whose preimage can be written as a disjoint union of open sets in $\tilde{X}$,

$$p^{-1}(U) = \bigsqcup_{i \in I} V_i,$$

such that the restrictions $p\restriction_{V_i}: V_i \to U$ are homeomorphisms for all $i \in I$.
Note: We allow the union to be empty, so we do not require p to be surjective; this is how it is defined in Hatcher's book.

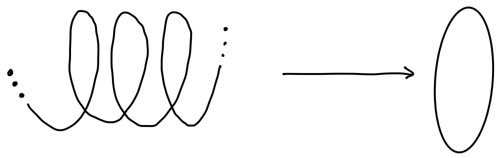
Example 5.12.43. Let $X = S^1$ be the one dimensional circle. We give three examples of covering spaces.

1. $\tilde{X} = S^1 \sqcup S^1$ with p acting 'identically' on each of the components. We can see this as two circles directly above our space X with p being the projection down.



2. Instead of taking two circles, we can take just one circle, and deform it so that it goes around twice above X; we get $\tilde{X} = S^1$, and p sending $s(\alpha)$ to $s(2\alpha)$, where $s : [0, 2\pi] \to S^1$ is the parametrization of S^1 by angle.

3. We can take a spiral above our circle X, with projection down. We can realize this as $\tilde{X} = \mathbb{R}$, and $p : x \mapsto s(2\pi x)$, for s the same as in the previous example.



Definition 5.12.44 (Fundamental group)**.** The *fundamental group* of a topological space X with a basepoint x_0, denoted by $\pi_1(X; x_0)$, is the set of all homotopy classes $[f]$ of loops $f : [0, 1] \to X$ at the basepoint x_0, with composition as the binary operation.

Example 5.12.45. $\pi(S_1, x_0) = \mathbb{Z}$ for any $x_0 \in S^1$.

There is a nice correspondence between the topological concept of covering spaces, and the algebraic concept of fundamental group.

Theorem 5.12.46 (Classification of covering spaces, Hatcher pg. 67)**.** *Suppose that X is a path-connected*[25]*, locally path-connected*[26]*, and semilocally simply connected*[27] *topological space. Then there is a bijection between path-connected covering spaces $p : \tilde{X} \to X$ (up to isomorphism) and subgroups of the fundamental group $\pi_1(X, x_0)$ (up to conjugacy).*

Example 5.12.47. What are the subgroups of $\mathbb{Z}$ corresponding to the three examples in Example 5.12.43?

1. The covering space there is not connected. It is a disjoint union of two connected components. The theorem classifies those separately. One component is just S^1 with p being the identity, which corresponds to the whole fundamental group $\mathbb{Z}$ on the algebraic side.

2. This covering corresponds to the subgroup $2\mathbb{Z} \subseteq \mathbb{Z}$.

3. The universal covering corresponds to the trivial subgroup $0 \subseteq \mathbb{Z}$

[25] There is a continuous path between every pair of points.

[26] Each point has an arbitrarily small neighbourhood that is path-connectd. More precisely, for every $x \in X$ and open neighbourhood U of x, there exists an open neighbourhood $V \subseteq U$ of x which is path-connected. [Hatcher, pg 62]

[27] Each point $x \in X$ has a neighbourhood U such that the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by inclusion is trivial, i.e., sends every loop to the trivial loop. [Hatcher, pg 63]

5.12.2 From covering spaces to fiber bundles

Let us look at the local condition in the definition of a covering space of X (Def. 5.12.42). For each point $x \in X$, there needs to be small enough neighbourhood U so that $p^{-1}(U) = \bigsqcup_{i\in I} V_i$ is a disjoint union, where each V_i is homeomorphic to U via p. This condition is equivalent to saying that $\bigsqcup_{i\in I} V_i$ is homeomorphic to $U \times I$ where we take the discrete topology on the index set I. But what if we took some different topology on I? This is exactly what gets us to the notion of a *fiber bundle.*

Definition 5.12.48 (Fiber bundle)**.** A *fiber bundle* over a topological space X with *fiber* F is a topological space E, called a *total space*, together with a surjective continuous map

$$p : E \to X,$$

satisfying the following condition: for each $x \in X$ there exists a neighbourhood U of x with a homeomorphism

$$h : p^{-1}(U) \to U \times F.$$

which makes the diagram

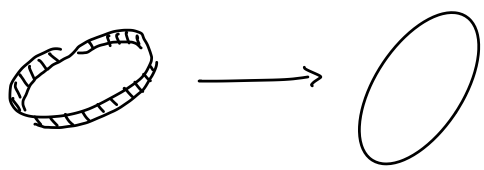
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\mathrm{Proj}_U"] \\ & U & \end{tikzcd}$$

commute.

Fiber bundles are 'locally product-like', but they do not need to be a product globally, as the following examples demonstrate.

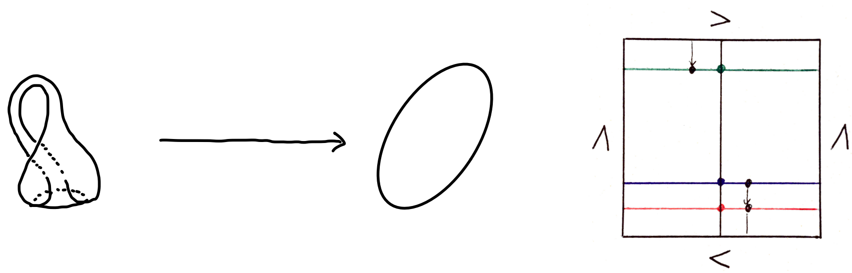
Example 5.12.49.

1. Möbius strip with projection on the 'middle circle' is a fiber bundle over S^1 with fibers $[0,1]$. For small open sets U we have $p^{-1}(U) \cong U \times [0,1]$.



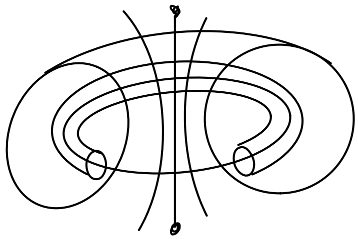
   Compare this just to taking a cylindrical surface $S^1 \times [0,1]$ with a projection to $S^1 \times \{\frac{1}{2}\}$. Locally, this fiber bundle looks the same, but the total spaces are clearly different.

2. Similarly, we can take Klein bottle with projection such that $p^{-1}(U) \cong U \times S^1$, and we get a fiber bundle over S^1 with fibers S^1. We can see this for example by taking the representation by square with identified sides – if we draw a straight line going through the middle of the square, we get a circle, and we can project vertically to that circle.



   We can do the same for torus, and get a fiber bundle which looks the same locally, but the total space is different – a product $S^1 \times S^1$ globally.

3. Hopf fibration is a fiber bundle over S^2 with fibers S^1 and total space S^3. That is, we have a projection $p : S^3 \to S^2$ such that for small neighbourhoods U, the preimage is $p^{-1}(U) \cong U \times S^1$. We will not define p here, but this construction gives a nice way to visualise S^3 in R^3.[28]



5.12.3 Moving towards sheaves.

What if the fiber F had some additional structure to topology, like a structure of a group or a vector space? For instance in the Example 5.12.43.3, with a spiral covering a circle, the fiber has naturally the additive structure of $\mathbb{Z}$ rather than being just a discrete topological space.

Another, more general example, is taking any fiber bundle, and replacing the fibers F with a (co)homology, so that locally we have projections $U \times H_i(F,k) \xrightarrow{\pi} U$. This changes the total space too. Taking the example of Klein bottle, we can apply $H_1(S^1, \mathbb{R})$, so that our 'fibers' are actually the vector space $\mathbb{R}$ – see Figure 8. We will not go into details at this point, but the cycle generating $\mathbb{R}$ in the homology group comes with some orientation; if we take a point $(u,1) \in U \times H_1(S^1, \mathbb{R})$, we can follow it along a closed curve going around the Klein bottle, and the sign of the second coordinate will flip by the time we get back to u. This is basically a reformulation of the fact that the Klein bottle is not orientable.

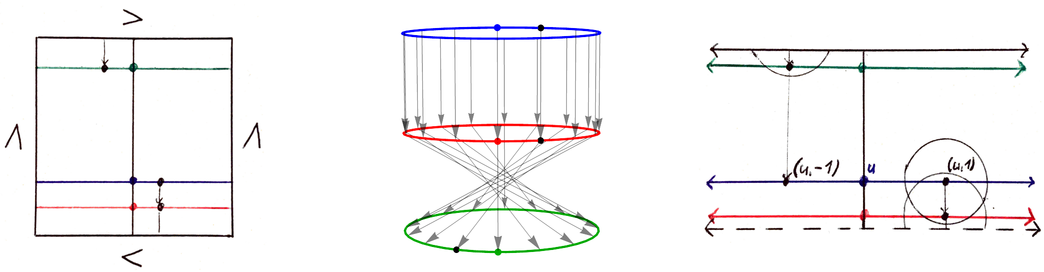


Figure 8: On the left, we see the Klein bottle as a fiber bundle over S^1, the vertical black circle, and with fibers also S^1, the horizontal circles in color. We apply $H_1(\cdot, \mathbb{R})$ to each of the horizontal circles, getting $\mathbb{R}$ as new fibers instead – this yields the space on the right picture, which looks like an "infinite Möbius strip". The maps between those new fibers are induced by the inclusion maps between the original fibers – as shown in the middle picture.

Assigning some algebraic structure to neighbourhoods of points is starting to look like sheaf theory – and that is where we are heading. In particular, this is the *étale space* perspective of sheaf theory.

Definition 5.12.50 (Étale space)**.** The *étale space* of a presheaf F on a topological space X is the disjoint union

$$\mathrm{Et}(F) = \bigsqcup_{x \in X} F_x,$$

together with the topology generated by open sets of the form

$$U_{(s,V)} = \{ s_y \in F_y \,|\, y \in V \},$$

[28]See, e.g., here.

for each pair (s,V), $s \in F(V)$, V open subset of X. Recall that s_y denotes the *germ* of s over y, which is the image of s under the restriction map

$$\begin{aligned} F(V) &\to F_y \\ s &\mapsto s_y, \end{aligned}$$

where $F_y = \varinjlim_{W \ni y} F(W)$ is the *stalk* at y.

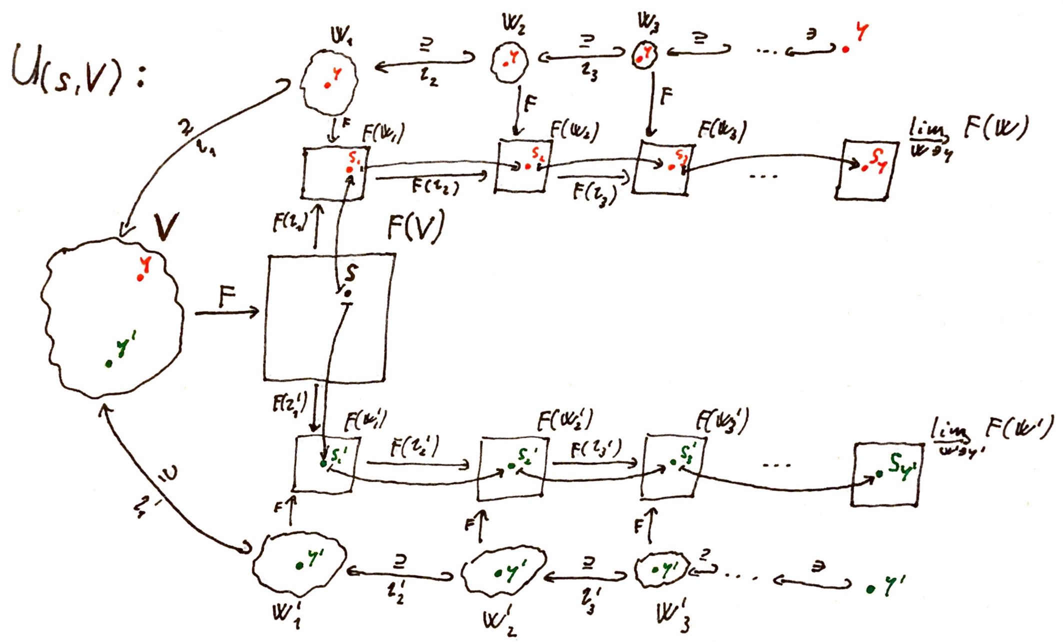


Figure 9: The open set $U_{(s,V)}$ in the étale space for an open set $V \subseteq X$ and an element $s \in F(V)$ is the set of all germs s_y over points $y \in V$. In this scheme we see how we get to two particular elements $s_y, s_{y'} \in U_{(s,V)}$.

The étale space has several very nice properties. For X, F and $\mathrm{Et}(F)$ as in the definition, we define the projection $p : \mathrm{Et}(F) \to X$ by sending an element $e \in F_y$ to y. Then we have the following:

1. $\mathrm{Et}(F)$ is a topological space,
2. p is a local homeomorphism onto X,
3. $p^{-1}(x) = F_x$ is an abelian group,
4. group operations $+_x : F_x \times F_x \to F_x$ are continuous,
5. the functor defined by $F^+(U) \coloneqq \{\text{continuous sections } s : U \to \mathrm{Et}(F)\}$ is the sheafification (see Definition 5.10.11) of F.

Let us look at some examples of constructing étale spaces and sheafifications out of them.

Example 5.12.51.

1. Consider the constant presheaf: we take any topological space X, and the functor F sending all open sets to $\mathbb{Z}$, and all morphisms to identity. What is $\mathrm{Et}(F)$?

   For every $x \in X$, the stalk over x is $F_x = \mathbb{Z}$, so as a set, $\mathrm{Et}(F) = \bigsqcup_{x \in X} \mathbb{Z}$, which we can realize as $\mathbb{Z} \times X$. We need to compute the open sets generating the topology of $\mathrm{Et}(F)$. Let $V \subseteq X$ be open, and let $n \in F(V) = \mathbb{Z}$. For any $y \in V$, the germ $n_y \in F_y$ is n, because the restriction map $F(V) \to F_y$ is the identity $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$. And if we view F_y as part of $\mathrm{Et}(F) = \mathbb{Z} \times X$, we have $n_y = (n, y)$. Hence, the open set we get for the pair (n, V) is

$$U_{(n,V)} = \{n_y \in F_y \,|\, y \in V\} = \{(n,y) \,|\, y \in V\} = \{n\} \times V.$$

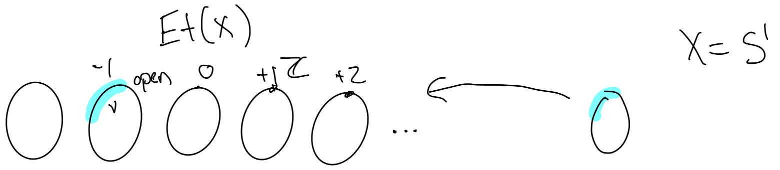
The topology on $\mathrm{Et}(F)$ are all finite intersections and arbitrary unions of these sets. We end up with $\mathrm{Et}(F)$ being $\mathbb{Z}\times X$ even as a topological product with discrete topology on $\mathbb{Z}$.

Now sheafification of F is F^{+} given by

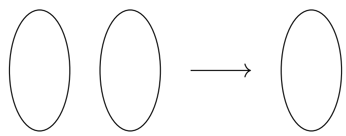
$$F^{+}(U) \coloneqq \{s : U \to \mathbb{Z}\times X \mid s \text{ continuous section}\}.$$

A continuous section $s : U \to \mathbb{Z}\times X$ with respect to the projection $p : \mathbb{Z}\times X \to X$ sends each connected component V of U identically to $\{n\}\times V$ for some n. Now if we put the additive structure of the group on the set of sections by adding the first coordinates, we have $F^{+}(U) \cong \mathbb{Z}^{(\#\text{ of connected components of } U)}$.

The figure bellow shows $\mathrm{Et}(F)$ in the case $X = S^1$.



2. In the previous example we have seen how to construct covering of S^1 by infinitely many disjoint circles as an étale space – we started with constant presheaf sending everything to $\mathbb{Z}$. We can use exactly the same construction to get a covering by any number of disjoint circles when we start with constant presheaf sending everything to an abelian group A of the desired cardinality. For example for $\mathbb{Z}_2$ we get covering by two circles.



Topologically, the étale space only depends on the cardinality of A, but étale space has an additional abelian structure, and so we get different étale spaces for A being $\mathbb{Z}_4$ and $\mathbb{Z}_2\times\mathbb{Z}_2$.

3. We would like to construct étale space E on S^1 which would topologically contain the covering $\tilde{X}$ by the twice-twisted circle as in Example 5.12.43, 2. We need to start with a presheaf more complicated than a constant one, otherwise we would get something analogous to the previous examples. But where should we even start looking? The answer is: the correspondence of étale spaces and sheaves!

We aim to obtain étale space whose continuous sections are the sections of the twisted-circle covering $\tilde{X}$. We can naïvely define a presheaf by

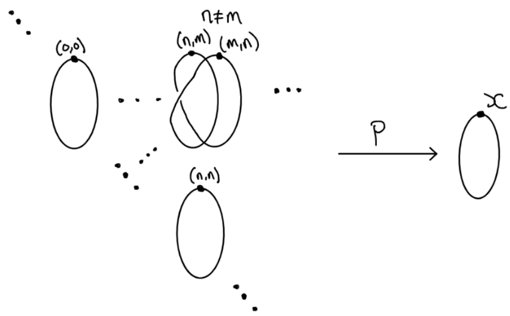
$$\tilde{F}(U) = \left\{s : U \to \tilde{X} \,\middle|\, s \text{ is a continuous section}\right\}.$$

However, no structure of an Abelian group is given to $\tilde{F}(U)$. Moreover, $\tilde{F}(S^1) = \varnothing$, which can not even be given any group structure. But there is a simple trick to create Abelian groups from any set – we can take the free Abelian group generated by the sections instead of the sections themselves. So we define a presheaf F by

$$F(U) = \mathbb{Z}^{\tilde{F}(U)} = \mathbb{Z}^{\left\{s:U\to\tilde{X} \,\middle|\, s \text{ is a continuous section}\right\}}.$$

This is still not a sheaf, but we can now define étale space $\mathrm{Et}(F)$, and then we get the sheafification of F by taking continuous sections of the étale space.

For every point of $y \in S^1$, if we take a small enough neighbourhood U, then $F(U) = \mathbb{Z}\times\mathbb{Z}$ — we get one copy of $\mathbb{Z}$ for each of the two sections. Since this is also true for all neighbourhoods $V \subseteq U$, we still have the same group in the colimit, that is, $F_y = \mathbb{Z}\times\mathbb{Z}$. This means that as a set we have $\mathrm{Et}(F) = \mathbb{Z}\times\mathbb{Z}\times S^1$, which we can see as an infinite matrix of circles. But the topological structure on this set is very different than the usual topology on $\mathbb{Z}\times\mathbb{Z}\times S^1$. Without a proof, let us just state that in he étale topology, the circles (m, n, S^1) and (n, m, S^1) are connected so that they produce the twisted double-circle $\tilde{X}$.



5.12.4 Local systems and constant sheaves

Definition 5.12.52 (Constant sheaf)**.** A sheaf F on X is called *constant* if there exists a sheaf G on $\{\text{pt}\}$ such that

$$p^*G \cong F,$$

where $p : X \to \{\text{pt}\}$ is the map sending everything to one point.

Exercise 5.12.53 (easy)**.** *Show that F is a constant sheaf iff it is the sheafificaton of a presheaf of the form $\mathbb{1}_G$, i.e., a presheaf which assigns the same group/vector space/... G to every open set, and has identities as all the restriction maps.*

Definition 5.12.54 (Locally constant sheaf)**.** A sheaf F on X is *locally constant* (or a *local system* if F is valued in $\mathbf{Vect}_k$) if for each $x \in X$ there exists a neighborhood U of x such that

$$i_U^*F \text{ is a constant sheaf,}$$

where $i_U : U \hookrightarrow X$. This means that we have some sheaf G on $\{\text{pt}\}$ such that

$$\begin{tikzcd} \{\text{pt}\} & U \arrow[l, "p"'] \arrow[r, hook, "i_U"] & X \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, squiggly, "p^*"] & p^*G \cong i_U^*F & F \arrow[l, squiggly, "i_U^*"'] \end{tikzcd}$$

Note that we often denote i_U^*F by $F{\restriction_U}$.

Example 5.12.55. The sheaf we get from the étale space in Example 5.12.51, 3., is not constant, but it is locally constant.

Analogous to the connection between covering spaces and the fundamental group, there is a connection between local systems (locally constant sheaves valued in $\mathbf{Vect}_k$) and the representations of the fundamental group.

Theorem 5.12.56. *Suppose that X is a path-connected, locally path-connected, semilocally simply connected, and locally compact*[29] *topological space. Then there is a bijection (even an equivalence of categories)*

$$\{\textit{local systems on } X\}\big/\{\textit{isomorphisms}\} \longleftrightarrow \{\textit{representations of } \pi_1(X, x_0)\}\big/\{\textit{isomorphisms}\}$$

In what follows we will describe a large family of well-behaved sheaves, which in a certain sense are 'built' out of locally constant pieces. Sheaves in this family will be called constructible, and will provide a long list of well-behaved examples for which we can compute sheaf cohomology.

5.12.5 Σ-constructability

If we have a triangulation $h : \Sigma \to X$ of a space X, where Σ is an (abstract) simplicial complex, then for $\sigma \in \Sigma$, $h(\sigma)$ is homeomorphic to σ, which is a very well-behaved space, and so the local systems on $h(\sigma)$ are easy to describe. This can be very useful for calculations, because reasonable simplicial complexes are much easier to work with than general topological spaces.

[29]Space X is *locally compact* if for every $x \in X$ there exists a neighborhood U of x and a compact $K \subseteq X$ such that $U \subseteq K$.

Definition 5.12.57 (Σ-constructible)**.** A sheaf F on X is Σ-*constructible* (or Σ-*cellular*) if

- $h : \Sigma \to X$ is a triangulation of X,
- $F\restriction_{h(\sigma)}$ is a constant sheaf for every $\sigma \in \Sigma$.

For a Σ-constructible sheaf, the constant sheaf on a simplex $\sigma \in \Sigma$ is fully described by one abelian group. This gives us the following proposition.

Proposition 5.12.58. *For $h : \Sigma \to X$, a fixed triangulation of a topological space, there is an equivalence of categories*

$$\Sigma\text{-}\textit{constructible sheaves on } X \longleftrightarrow \mathrm{Fun}(\Sigma, \mathbf{Ab}),$$

where Σ is viewed as a poset category.

Exercise 5.12.59. *Find the two functors realising the category equivalence in Proposition 5.12.58.*

This approach can be used to compute the sheaf cohomology of the sheaf from Example 5.12.51.3. We can triangulate $X = S^1$ by a triangle Σ so that F is Σ-constructible, and then do the calculation in $\mathrm{Fun}(\Sigma, \mathbf{Ab})$.

Lecture 13: Cohomology

5.13.1 Čech cohomology

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Define the ordered set of $(n+1)$ tuples of indexes:

$$I^n := \{(\alpha_0, \dots, \alpha_n) \in I^{n+1} : U_{\alpha_0} \cap \dots \cap U_{\alpha_n} \neq \varnothing\}.$$

With this we can construct the Čech complex begining with it's chain groups:

$$\check{C}^n(\mathcal{U}; k) := \{f : I^n \to k\},$$

for k some field or ring. Further, define the differential between the chain groups of the Čech complex:

$$\begin{aligned} d_n : \check{C}^n(\mathcal{U}; k) &\to \check{C}^{n+1}(\mathcal{U}; k) \\ f &\mapsto d_n f((\alpha_0, \dots, \alpha_n)) := \sum_{i=0}^{n+1} (-1)^i f((\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{n+1})), \end{aligned}$$

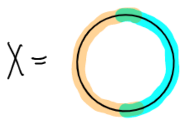
where the the hatted ith index is removed in each summand. Finally taking the quotient of kernels and images of codimension 1 differentials we obtain the Čech cohomology of $\mathcal{U}$:

$$\check{H}^n(\mathcal{U}, k) := {}^{\ker d_n}\!\big/_{\operatorname{im} d_{n-1}}.$$

Additionally, Čech chomology is exactly the simplicial cohomology of the nerve of the open cover $\mathcal{U}$.

Examples of Čech cohomology

1. Let $X = S^1$ and $\mathcal{U} = \{U_1, U_2\}$.



We first compute the Čech complex:

$$\begin{aligned}\check{C}^0(\mathcal{U};k) &= \{f : \{1,2\} \to k\}\\ &\simeq k \oplus k\\ \check{C}^1(\mathcal{U};k) &= \{f : \{(1,2)\} \to k\}\\ &\simeq k\\ d_0 f((1,2)) &= f(1) - f(2),\end{aligned}$$

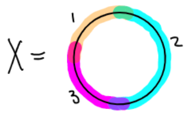
which may be written as $k^2 \xrightarrow{d_0} k$, from which we compute $\check{H}^0(\mathcal{U},k) \cong k$ and $\check{H}^1(\mathcal{U},k) \cong 0$. Unfortunately we do not recover the expected cohomology of the circle, however, we recall that the Čech cohomology is really the simplicial cohomology of the nerve of the cover, which in this instance is identical to that of a line.

Definition 5.13.60. An open cover $\mathcal{V}$ *refines* $\mathcal{U}$ if each V_j is contianed in U_i for some $i \in I$. Refinements give the set of all open covers of a space X the structure of a category, $\mathcal{U} \to \mathcal{V}$ if $\mathcal{V}$ refines $\mathcal{U}$.

Definition 5.13.61. The Čech cohomology of a space X is given by the colimit,

$$\check{H}^i(X;k) = \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U};k).$$

2. Let $X = S^1$ and $\mathcal{V} = \{U_1, U_2, U_3\}$ (i.e. $\mathcal{V}$ refines $\mathcal{U}$).



Again we compute the Čech complex:

$$\begin{aligned}\check{C}^0(\mathcal{V};k) &= \{f : \{1,2,3\} \to k\}\\ &\simeq k \oplus k \oplus k\\ \check{C}^1(\mathcal{V};k) &= \{f : \{(1,2),(1,3),(2,3)\} \to k\}\\ &\simeq k \oplus k \oplus k\\ d_0((\alpha_i,\alpha_j)) &= f(i) - f(j),\end{aligned}$$

which may be written as $k^3 \xrightarrow{d_0} k^3$. Now we require the matrix representation of d.

$$d_0 = \begin{pmatrix} 1 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}$$

From this we can compute $\check{H}^0(\mathcal{V},k) = k(1,1,1) \simeq k$ and $\check{H}^1(\mathcal{V},k) \cong k$. Which does recover the expected circle cohomology. In fact tkaing the colimit we obtain $\check{H}^0(X,k) = k$ and $\check{H}^1(X,k) = k$.

5.13.2 Čech cohomology for presheaves

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and define I^n as before:

$$I^n \coloneqq \{(\alpha_0, \ldots, \alpha_n) \in I^{n+1} : U_{\alpha_0} \cap \ldots \cap U_{\alpha_n} \neq \varnothing\},$$

and additionally let F be a presheaf on X. Now define the Čech complex with coefficients in F.

$$\check{C}^n(\mathcal{U},F) \coloneqq \prod_{(\alpha_0,\ldots,\alpha_n)\in I^n} F(\cap_{\alpha_i} U_{\alpha_i})$$
$$d_n : \check{C}^n(\mathcal{U},F) \to \check{C}^{n+1}(\mathcal{U},F)$$

Tracking the image of the differential is a little tricky. If $\beta = \{\beta_0,\ldots,\beta_n\} \subseteq \{\alpha_0,\ldots,\alpha_{n+1}\} = \alpha$ then there exists i such that $\beta = (\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{n+1})$. Now since the intersection of α sets intersects more sets it must be a subset of the intersection of β sets, so the map $R_{\alpha_i} : F(U_{\beta_0},\ldots,U_{\beta_n}) \to F(U_{\alpha_0} \cap \ldots \cap U_{\alpha_{n+1}})$ is a restriction.

Now we can construct a differential where we consider all the possible ways we could insert an extra term.

$$F(U_{\beta_0},\ldots,U_{\beta_n}) \xrightarrow{d_{(\beta_0\cap\ldots\cap\beta_n)}} \prod_{\substack{\alpha\in I^{n+1} \\ \text{s.t. } \beta\subseteq\alpha}} F(U_{\alpha_0}\cap\ldots\cap U_{\alpha_{n+1}})$$
$$d_{(\beta_0\cap\ldots\cap\beta_n)} \coloneqq \sum_i (-1)^i R_{\alpha_i}$$

And finally we obtain the full differential which is just the combination of all the d_β

$$d_n \coloneqq \prod_{\beta\in I^n} d_\beta.$$

We are now ready to compute the presheaf cohomology of $\mathcal{U}$ and X.

Definition 5.13.62. $\check{H}^n(\mathcal{U};F) \coloneqq {}^{\ker d_n}\!/\!{}_{\operatorname{im} d_{n-1}}$, $\check{H}^n(X;F) \coloneqq \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U};F)$.

Definition 5.13.63. A topological space X is paracompact if for each open cover $\mathcal{U}$ there exists a locally finite open refinement $\mathcal{V}$.

Now we additionally assume X is paracompact and F is a sheaf. Recall Definition 4.9.11 for sheaves. This means for any open cover $\mathcal{U}$ the sequence

$$0 \to F \to \check{C}^0(\mathcal{U};F) \to \check{C}^1(\mathcal{U};F) \to \ldots$$

is a resolution of F (not necessarily injective).

Theorem 5.13.64. *Assume X is Hausdorff and paracompact, F is a presheaf on X, and let F^+ be the sheafification of F. Then for all $n \geq 0$,*

$$\check{H}^n(X;F) \simeq \check{H}^n(X;F^+) \simeq R^n\Gamma(X;F^+).$$

This theorem is useful for two reasons, we can use the relatively simpler Čech cohomology to compute sheaf cohomology, and we do not need to go through the sheafification process. Without going into the proof we can observe that presheaves and sheaves agree on small local scales but disagree globally and the limit is shrinking the cover elements and using tiny restriction maps to achieve the correspondence.

5.13.3 Cellular Sheaf Cohomology

Assume $\Sigma \xrightarrow{h} X$ is a triangulation, and F is Σ-constructible, with respect to Σ (i.e. further shrinking cover elements wont change F). Define $\mathrm{St}(\sigma) \coloneqq$ the open star of $\sigma = \{\tau : \tau \geq \sigma\}$. We have an open cover of X consisting of open stars:

$$\mathcal{U} = \{h(\mathrm{St}(\sigma))\}_{\sigma\in\Sigma_0}.$$

Moreover, the nerve of $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$, is homeomorphic to Σ (i.e. $\mathcal{N}(\mathcal{U}) \simeq \Sigma$).

Now we may compute the Čech cohomology relative to $\mathcal{U}$:

$$0 \to \check{C}^0(\mathcal{U};F) \to \check{C}^0(\mathcal{U};F) \to \ldots$$

Unpacking this chain we obtain the regular sequence of increasing numbers of intersections of open sets:

$$0 \to \prod_{\sigma \in \Sigma_0} F(h(\mathrm{St}(\sigma))) \to \prod_{(\sigma,\tau)} F(h(\mathrm{St}(\sigma)) \cap h(\mathrm{St}(\tau))) \to \dots$$

This can be simplified further as the increasing intersections of the open sets is exactly the simplices of increasing dimension:

$$0 \to \prod_{\sigma \in \Sigma_0} F(h(\mathrm{St}(\sigma))) \to \prod_{\sigma \in \Sigma_1} F(h(\mathrm{St}(\sigma))) \to \prod_{\sigma \in \Sigma_2} F(h(\mathrm{St}(\sigma))) \to \dots$$

We can further refine this using Proposition 5.12.58:

$$\Sigma\text{-constructible sheaves} \leftrightarrow \text{Functors on } \Sigma$$

This lets us think of F just as a functor from Σ to $\mathbf{Ab}$. The resolution then becomes:

$$0 \to \prod_{\sigma \in \Sigma_0} F(\sigma) \to \prod_{\sigma \in \Sigma_1} F(\sigma) \to \prod_{\sigma \in \Sigma_2} F(\sigma) \to \dots$$

Again we still need to define the differential map, however, it is analogous to the Čech cohomology differential. We are finally ready to define simplicial sheaf cohomology.

Definition 5.13.65.

$$H^n(\Sigma; F) \coloneqq {}^{\ker d_n} \big/ {}_{\operatorname{im} d_{n-1}}.$$

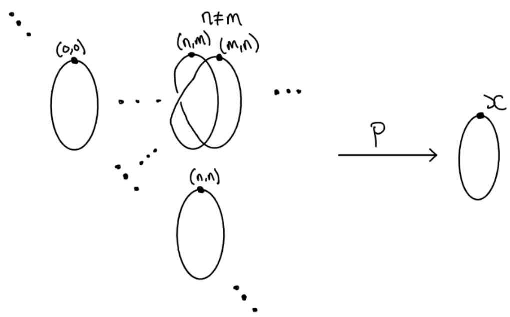
Theorem 5.13.66.

$$H^n(\Sigma; F) \cong R^n\Gamma(X; F)$$

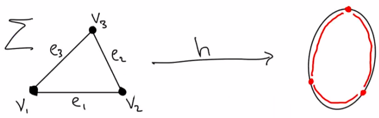
This sequence of simplifications is dependent on the initial triangulation being constructible with respect to F, which is the most difficult step in the process.

Examples of simplicial sheaf cohomology

1. What is the simplicial sheaf cohomology of the projection map:



We begin by fixing a triangulation of X.



$F(\sigma) \cong \mathbb{Z} \oplus \mathbb{Z}$ for all $\sigma \in \Sigma$, $F(\sigma \to \tau) = \mathrm{id}$ for all $(\sigma \to \tau) \neq (v_3 \to e_3)$, and instead

$$\begin{aligned} F(v_3 \to e_3) : \mathbb{Z} \oplus \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z} \\ (n, m) &\mapsto (m, n) \end{aligned}$$

Now we can write down the complex as:

$$0 \to \mathbb{Z}^6 \xrightarrow{d} \mathbb{Z}^6 \to 0.$$

The differential, d, is then given by the matrix

$$\begin{array}{cc} & \begin{array}{cccccc} v_1 & & v_2 & & v_3 & \end{array} \\ \begin{array}{c} e_1 \\ \\ e_2 \\ \\ e_3 \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix} \end{array}$$

Each one and two cell is assigned to two copies of $\mathbb{Z}$ and the maps between them are either $\pm\mathrm{id}$ or 0, except for the lower right entry which as we noted was the map switching the indices. From this we compute $H^0(X;F) \simeq \mathbb{Z}(1,1,1,1,1,1)$, $H^1(X;F) \simeq \mathbb{Z}^6/(a,b,c,d,e,-e) \simeq \mathbb{Z}\oplus\mathbb{Z} \big/ (n,m) \sim (m,n)$. If we were just considering the ranks of these groups this would coincidentally be equivalent to the cohomology of the circle. The 0th cohomology of the sheaf is really counting how many global sections we have and the digonal is the only place we can get a continuous section which is why we get $\mathbb{Z}$. Then by looking at the double loop we see 1st cohomology is $\mathbb{Z}\oplus\mathbb{Z}$ but we have to also consider the that the points (n,m) and (m,n) are really sitting on the same circle so intuitively we get the desired quotient. However, this is certainly calculating something different to regular cellular homology.

2. In general, for $X = S^1$ and Σ as before

$$\begin{array}{ccccc} \mathbb{Z} = \text{Rep. of } \Pi_1(X,x_0) & \longleftrightarrow & \text{Local System } \mathcal{L} \text{ on } X & \longrightarrow & \Sigma\text{-constructible sheaf on } X \\ \updownarrow & & & & \updownarrow \\ M : V \to V & \dashrightarrow & & \dashrightarrow & \mathrm{Fun}(\Sigma, \mathbf{Vect}) \end{array}$$

We start with representations of the fundemental group, which is simply $\mathbb{Z}$. Representations of $\mathbb{Z}$ may be packaged as a square matrix M, giving the bijection on the left. We can also compute a local system $\mathcal{L}$ from our representation which can also be transformed into a Σ constructible sheaf for our triangulation. This sheaf can of course be considered a functor from the triangulation to a vector space. So the entire pipeline can be considered to take a square matrix and return a functor. If we were to do this process, each simplex σ will be sent to the vector space, i.e. $F(\sigma) = V$; all the maps are sent to the identity apart from one, i.e. $F(\sigma \to \tau) = \mathrm{id}$ and $F(\gamma \to \lambda) = M$.

We can write this down as a complex:

$$0 \to V^3 \xrightarrow{d} V^3 \to 0$$

with boundary matrix:

$$d = \begin{pmatrix} \mathrm{id}_V & -\mathrm{id}_V & 0 \\ 0 & \mathrm{id}_V & -\mathrm{id}_V \\ \mathrm{id}_V & 0 & -M \end{pmatrix}$$

Now we are able to compute the cohomology:

$$H^0(X,\mathcal{L}) \simeq \ker d \cong +1 \text{ eigenspace of } M$$

$$H^1(X,\mathcal{L}) \simeq \text{coinvariants of } M = V \big/ (V - M_V)$$

These sheaf cohomology groups are potentially much more complicated than the standard cohomology of X so we have obtained an interesting generalisation of cohomology. In fact, we could repeat this computation without ever knowing what the fundemental group was by replacing the boundary maps with selected matrices and setting up the complex in the same way.

The reader should make a close comparison of the example above with Example 2.4.28, Example 2.5.40, and Example 3.7.21. The idea is to at least partially bring this journey full-circle, so to speak.

6 Notes, remarks

6.1 Additional definitions and lemmas

6.1.1 Composing natural transformations

There are two ways in which it makes sense to compose natural transformations. They are called *vertical* and *horizontal* composition.

Vertical composition. The vertical composition is very straightforward. Let F, G, H be three functors between fixed categories $\mathcal{A}$ and $\mathcal{B}$, and let us consider two natural transformations, σ from F to G, and τ from G to H. Then we can obtain a natural transformation $\tau \circ \sigma : F \Rightarrow H$ by taking morphisms $(\tau \circ \sigma)_X \coloneqq \tau_X \circ \sigma_X$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{B}
\arrow[from=F, to=G, Rightarrow, "\sigma"]
\arrow[from=G, to=H, Rightarrow, "\tau"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "\sigma_X"] \arrow[d, "Ff"] \arrow[rr, bend left, "(\tau\circ\sigma)_X"] & G(X) \arrow[r, "\tau_X"] \arrow[d, "Gf"] & H(X) \arrow[d, "Hf"] \\
F(Y) \arrow[r, "\sigma_Y"] \arrow[rr, bend right, "(\tau\circ\sigma)_Y"'] & G(Y) \arrow[r, "\tau_Y"] & H(Y)
\end{tikzcd}
$$

Horizontal composition. The horizontal composition is a bit trickier. Assume we have three categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$, pairs of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and $G, G' : \mathcal{B} \to \mathcal{C}$, and natural transformations $\sigma : F \Rightarrow F'$ and $\tau : G \Rightarrow G'$. We would like to construct a natural transformation ψ = '$\tau \circ \sigma$' from $G \circ F$ to $G' \circ F'$.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "F'"'{name=F2}] & \mathcal{B} \arrow[r, bend left, "G"{name=G}] \arrow[r, bend right, "G'"'{name=G2}] & \mathcal{C}
\arrow[from=F, to=F2, Rightarrow, "\sigma"]
\arrow[from=G, to=G2, Rightarrow, "\tau"]
\end{tikzcd}
$$

We can construct $\psi : GF \Rightarrow G'F'$ in two natural ways. We can go from GF to $G'F'$ via GF' or via $G'F$. For the first option, look at the diagram bellow. Starting with two objects and a morphism in $\mathcal{A}$, on the left, we use the naturality of σ, and get the commutative diagram in the middle. We apply the functor G to this diagram, which yields, again, a commutative diagram — the left square in the third diagram. To get the right commutative square in that diagram, we consider $F'(X) \xrightarrow{F'f} F'(Y)$, and apply the naturality of τ.

$$
\begin{tikzcd}
A \arrow[d, "f"] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "\sigma_X"] \arrow[d, "Ff"] & F'(X) \arrow[d, "F'f"] \\
F(Y) \arrow[r, "\sigma_Y"] & F'(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
GF(X) \arrow[r, "G\sigma_X"] \arrow[d, "GFf"] & GF'(X) \arrow[r, "\tau_{F'(X)}"] \arrow[d, "GF'f"] & G'F'(X) \arrow[d, "G'F'f"] \\
GF(Y) \arrow[r, "G\sigma_Y"] & GF'(Y) \arrow[r, "\tau_{F'(Y)}"] & G'F'(X)
\end{tikzcd}
$$

Together the last diagram shows that if we define $\psi_X \coloneqq \tau_{F'(X)} \circ G\sigma_X$ for each $X \in \operatorname{Ob} \mathcal{A}$, we get a natural transformation $\psi : GF \Rightarrow G'F'$ as desired.

The second approach is very similar. We start exactly the same, but we apply G' to the middle diagram, which yields the right square in the last diagram. The left square is then the naturality of τ applied to the morphism $F(X) \xrightarrow{Ff} F(Y)$.

$$\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\sigma_X"] \arrow[d, "Ff"] & F'(X) \arrow[d, "F'f"] \\ F(Y) \arrow[r, "\sigma_Y"] & F'(Y) \end{tikzcd} \qquad \begin{tikzcd} GF(X) \arrow[r, "\tau_{F(X)}"] \arrow[d, "GFf"] & G'F(X) \arrow[r, "G'\sigma_X"] \arrow[d, "G'Ff"] & G'F'(X) \arrow[d, "G'F'f"] \\ GF(Y) \arrow[r, "\tau_{F(Y)}"] & G'F(Y) \arrow[r, "G'\sigma_Y"] & G'F'(X) \end{tikzcd}$$

Again, the last diagram shows that if we define $\tilde{\psi}_X \coloneqq G'\sigma_X \circ \tau_{F(X)}$, we get a natural transformation $\tilde{\psi} : GF \Rightarrow G'F'$.

Now the question is whether the two approaches gives us the same natural transformation. We ask whether $\tau_{F'(X)} \circ G\sigma_X = G'\sigma_X \circ \tau_{F(X)}$ for each object X of $\mathcal{A}$, that is, whether the following diagram commutes:

$$\begin{tikzcd} GF(X) \arrow[r, "\tau_{F(X)}"] \arrow[d, "G\sigma_X"] & G'F(X) \arrow[d, "G'\sigma_X"] \\ GF'(X) \arrow[r, "\tau_{F'(X)}"] & G'F'(X) \end{tikzcd}$$

But this diagram is commutative due to naturality of τ applied to $F(X) \xrightarrow{\sigma_X} F'(X)$. Therefore, we can define the composition of σ and τ in either of those ways.

6.2 Examples, Remarks

Lecture 10: Example where the 'naïve image' of a sheaf morphism is not a sheaf

In Lecture 10 we have discussed that even though both $\mathrm{Fun}(\mathbf{Open}(\mathrm{X}), \mathbf{Ab})$, the category of presheaves, and $\mathbf{Shv}(\mathrm{X})$, the category of sheaves, are abelian, the images do not coincide – the image of a natural transformation between two sheaves in the category of presheaves might not be a sheaf. We need to take the sheafification of that to get the image in the category of sheaves.

Here is a simple example of a natural transformation between two sheaves whose image (as defined in the category of presheaves) is not a sheaf. Let us consider topological space $(X; \{X, U, V, W = U \cap V, \varnothing\})$, and two sheaves as on the following two figures:

$$\begin{tikzcd} & X \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & V \arrow[dl, dash] \\ & W \arrow[d, dash] & \\ & 0 & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\ \mathbb{Z} \arrow[dr, "\mathrm{id}"'] & & \mathbb{Z} \arrow[dl, "\mathrm{id}"] \\ & \mathbb{Z} \arrow[d] & \\ & 0 & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, "\mathrm{Proj}_1"'] \arrow[dr, "\mathrm{Proj}_2"] & \\ \mathbb{Z} \arrow[dr, "0"'] & & \mathbb{Z} \arrow[dl, "0"] \\ & \mathbb{Z} \arrow[d] & \\ & 0 & \end{tikzcd}$$

We define a natural transformation φ between the two sheaves:

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[rrr, "\Delta: x \mapsto (x,x)"] \arrow[dl, "\mathrm{id}"'] \arrow[ddr, "\mathrm{id}"] & & & \mathbb{Z} \oplus \mathbb{Z} \arrow[dl, "\mathrm{Proj}_1"'] \arrow[ddr, "\mathrm{Proj}_2"] & \\
\mathbb{Z} \arrow[rrr, "\mathrm{id}"] \arrow[ddr, "\mathrm{id}"'] & & & \mathbb{Z} \arrow[ddr, "0"] & & \\
 & & \mathbb{Z} \arrow[rrr, "\mathrm{id}"] \arrow[dl, "\mathrm{id}"] & & & \mathbb{Z} \arrow[dl, "0"] \\
 & \mathbb{Z} \arrow[rrr, "0"] \arrow[d] & & & \mathbb{Z} \arrow[d] & \\
 & 0 \arrow[rrr] & & & 0 &
\end{tikzcd}
$$

We can easily check that all the rectangles commute. The image of φ in the category of functors from $\mathbf{Open}(\mathrm{X})$ to $\mathbf{Ab}$ is the functor given by $\mathrm{im}(\varphi) : A \mapsto \mathrm{im}(\varphi_A)$ for each open set A in X. This is isomorphic to the following:

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & \\
\mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] \\
 & 0 \arrow[d] & \\
 & 0 &
\end{tikzcd}
$$

This is not a sheaf. The problem here is the (non-)existence of an element gluing local information from U and V to X—if we choose different element on U and on V, the restrictions commute, but there is no element for the union which would restrict to both at the same time. The problem is, that in the image, we lose the constraint given by W in the first sheaf. In the image, we can have two different elements for U and for V, and it is still true that their restrictions to W coincide, since those restrictions are the zero map.

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